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CALCULATION OF SYMMETRIC POLYNOMIALS IN THE MAPLE COMPUTER MATHEMATICS SYSTEM

Abstract. The article studies the theory of symmetric polynomials. As a result of the study, it was shown that one of the methods for solving systems of higher degree equations is the method using symmetric polynomials. The practical implementation of the solution method has a computer implementation in the Maple system.

Keywords: symmetric polynomials, higher degree equations, alternative methods of solving, analytical calculations.

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Introduction. Despite the variety of different theoretical aspects of algebra, the theory of polynomials occupies a special place. The theory is very meaningful. Symmetric polynomials, which have numerous applications, occupy a certain place in the theory of polynomials.

One of the many applications of symmetric polynomials is their successful use in solving systems of higher degree equations.

The development of the current direction of computer mathematics leads to questions of finding solutions to mathematical problems in modern packages of analytical calculations. As an alternative method, solving systems of higher degree equations in the Maple package is considered. The solution in the Maple package will speed up and simplify the implementation of the necessary mathematical actions, calculations and rationalize the solution process.

Conditions and methods of research. The solution of various geometry problems is often accompanied by the use of symmetry and its properties. When solving algebraic problems, this trend can also be traced. But in a semantic sense, the concepts of symmetry in geometry and algebra differ. The algebraic meaning is that the expression remains unchanged when the composition of its letters is rearranged. For example, expression x^2y^2z has symmetry with respect to variables x and y. But has not symmetry with respect to variables x and z. The result of the rearrangement x and y is the expression, a distinctive feature from the existing one is the order of factors y^2x^2z, the expression does not change. The result of the rearrangement x and z is the expression z^2y^2x, which is not identical in value to the Original.

Let us call a polynomial $p(x,y)$ in two variables x and y symmetric if, when replacing the first variable with the second variable, the polynomial does not change:

$$p(x,y) = p(y,x). \qquad (1)$$

[1] For example, polynomial $p(x,y) = x^2y + xy^2$ - symmetry:
$p(x,y) = x^2y + xy^2 = y^2x + yx^2 = p(y,x)$.
$g(x,y) = x^3 - 3y^2$ - does not apply to symmetrical:
$g(x,y) = x^3 - 3y^2 \neq y^3 - 3x^2 = g(y,x)$.

The simplest symmetric polynomials in variables mean the sum $x+y$: $\sigma_1 = x + y$ and product xy: $\sigma_2 = xy$.

Symmetric polynomials include the so-called power polynomials, the form of which $x^n + y^n$:

$$s_0 = x^0 + y^0 = 2,\ s_1 = x + y, s_2 = x^2 + y^2, \ldots, s_n = x^n + y^n\ . \qquad (2)$$

Let's study the method of forming symmetric polynomials. Into some asymmetric polynomial in $\sigma_1 = x + y$ and $\sigma_2 = xy$ - $p_n(\sigma_1, \sigma_2)$ let us replace the expressions σ_1 and σ_2 through x and y. It is obvious that a symmetric polynomial is formed from x and y. This happened because σ_1 and σ_2 they are not subject to changes when rearranged x and y. In this connection, the entire resulting polynomial, which is expressed by $x+y$ and xy. For example, let $p_3(\sigma_1, \sigma_2) = \sigma_1^3 - \sigma_1\sigma_2$. Using expressions $\sigma_1 = x + y$ and $\sigma_2 = xy$ we get the following [2]:

$$p_3(\sigma_1, \sigma_2) = \sigma_1^3 - \sigma_1\sigma_2 = (x+y)^3 - (x+y)xy = x^3 + 2x^2y + 2y^2x + y^3.$$

We will use this technique later. But first, let's look at the representation of power sums using σ_1 and σ_2 . We have:

$$s_1 = x + y = \sigma_1;$$
$$s_2 = x^2 + y^2 = (x+y)^2 - 2xy = \sigma_1^2 - 2\sigma_2;$$
$$s_3 = x^3 + y^3 = (x+y)(x^2 - xy + y^2) = (x+y)((x+y)^2 - 3xy) = \sigma_1(\sigma_1^2 - 3\sigma_2) = \sigma_1^3 - 3\sigma_1\sigma_2\ .$$

We got the representation of power sums s_1, s_2, s_3 in the form of polynomials σ_1 and σ_2.

Example 1.1. We have $P_4(x,y) = x^4 + 5x^3y + 6x^2y^2 + 5xy^3 + y^4$- symmetric polynomial in two variables. Let's express $P_4(x,y)$ from $\sigma_1 = x + y$ and $\sigma_2 = xy$. First, let's take steps to highlight expressions with explicit representations of power sums in terms of x, y [4]:

$$P_4(x,y) = x^4 + 5x^3y + 6x^2y^2 + 5xy^3 + y^4 = (x^4 + y^4) + 5xy(x^2 + y^2) + 6x^2y^2.$$

We use power sum formulas $s_2 = x^2 + y^2$, $s_4 = x^4 + y^4$ for the last expression of the polynomial $P_4(x,y)$ as a result we have:

$$P_4(x,y) = \underbrace{(x^4+y^4)}_{=S_4} + 5\underbrace{xy}_{=\sigma_2}\underbrace{(x^2+y^2)}_{=S_2} + 6\underbrace{x^2y^2}_{=\sigma_2^2}$$
$$= \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 5\sigma_2(\sigma_1^2 - 2\sigma_2) + 6\sigma_2^2 =$$
$$= \sigma_1^4 + \sigma_1^2\sigma_2 - 2\sigma_2^2.$$

Example 1.2 Solve a system of algebraic equations

$$\begin{cases} x^5 + y^5 = 33 \\ x + y = 3 \end{cases}.$$

The equations of the system contain symmetric polynomials of the 2th degree in x and y. Let's use the new unknowns:

$$\begin{cases} x + y = \sigma_1 \\ xy = \sigma_2 \end{cases}.$$

Using new unknowns σ_1 and σ_2, we obtain an expression for the equations of the system. Power sum S_5 - this is the left side of the first equation. S_5 can be expressed in terms of polynomials of variables σ_1 and σ_2 [5]:

$$x^5 + y^5 = s_5 = \sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1\sigma_2^2.$$

Let's do the same with the second equation of the system: we write the expression for the power sum S_1 in terms of polynomials in the variables σ_1 and σ_2. In our case, we have the following:

$$x + y = s_1 = \sigma_1.$$

Now let's write the system with new unknowns:

$$\begin{cases} \sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1\sigma_2^2 = 33 \\ \sigma_1 = 5 \end{cases}.$$

One unknown is already known: $\sigma_1 = 5$. Using it for the first equation of the system we obtain a quadratic equation for σ_2:

$$15\sigma_2^2 - 135\sigma_2 + 210 = 0,$$
$$\sigma_2^2 - 9\sigma_2 + 14 = 0.$$

We can easily find the roots of the equationσ_2:
$\sigma_2 = 2, \sigma_2 = 7.$

This means that to find the initial unknowns x and y you need to solve 2 systems of equations:

$$\begin{cases} x + y = 2 \\ xy = 7 \end{cases}, \begin{cases} x + y = 7 \\ xy = 2 \end{cases}.$$

The solution to the last 2 systems can be found without much effort:

$$\begin{cases} x_1 = 2 \\ y_1 = 1 \end{cases}, \begin{cases} x_2 = 1 \\ y_2 = 2 \end{cases}, \begin{cases} x_3 = \frac{3}{2} + \frac{\sqrt{19}}{2} i \\ y_3 = \frac{3}{2} - \frac{\sqrt{19}}{2} i \end{cases}, \begin{cases} x_4 = \frac{3}{2} - \frac{\sqrt{19}}{2} i \\ y_4 = \frac{3}{2} + \frac{\sqrt{19}}{2} i \end{cases}.$$

Using the above method, Bezout's theorem is often used to find solutions to systems of equations [1].

Research results. The process of solving systems of algebraic equations using symmetric polynomials is carried out by the substitution method. In order to use substitution, it is necessary to select the symmetric polynomial that will be used to implement the method.

One of the directions in the development of computing technologies is currently associated with the emergence and development of powerful mathematical packages that make it possible to simplify the process of preparing a problem, solving it and analyzing the results. Let's consider solving systems of algebraic equations in a computer mathematics system as an alternative solution method.

In Example 1.2, you need to solve a system of algebraic equations, the right side of which represents symmetric polynomials:

$$\begin{cases} x^5 + y^5 = 33 \\ x + y = 3 \end{cases}.$$

We implement the system solution in the Maple package. Enter the system equations and the solution command *solve* [3]:

restart;
sys:=({x^5+y^5=33,x+y=3});
s:=solve(sys,{x,y});

$$sys := \{x + y = 3, x^5 + y^5 = 33\}$$
$$s := \{x = 1, y = 2\}, \{x = 2, y = 1\},$$
$$\{x = -\mathrm{RootOf}(\_Z^2 - 3\_Z + 7) + 3, y = \mathrm{RootOf}(\_Z^2 - 3\_Z + 7)\}$$

The system provided a solution, but one of them included the *RootOf* function, which means that roots cannot be expressed in radicals. [3] Use the _EnvExplicit:=true option to get an explicit expression of the system roots:

restart;
_EnvExplicit:=true;
sys:=({x^5+y^5=33,x+y=3});
s1:=solve(sys,{x,y});

$$\_EnvExplicit := true$$
$$sys := \{x + y = 3, x^5 + y^5 = 33\}$$
$$s1 := \{x = 1, y = 2\}, \{x = 2, y = 1\}, \{x = \frac{3}{2} - \frac{1}{2} I \sqrt{19}, y = \frac{3}{2} + \frac{1}{2} I \sqrt{19}\},$$
$$\{x = \frac{3}{2} + \frac{1}{2} I \sqrt{19}, y = \frac{3}{2} - \frac{1}{2} I \sqrt{19}\}$$

The resulting solution corresponds to the solution obtained manually. In order for the system to check all solutions, we use the assignment command *assign(name)* [3]:

assign(s1[1]);x;y;

s2:=subs(s1[1],sys);

1

2

$s2 := \{3 = 3, 33 = 33\}$

As you can see, the first solution of the system is correct. In order to check all solutions at the same time, we use the *map* function together with the *subs* function:

map(subs,[s1],sys);

$[\{3 = 3, 33 = 33\}, \{3 = 3, 33 = 33\}, \{3 = 3, 33 = 33\}, \{3 = 3, 33 = 33\}]$

As you can see, the system has checked its solution.

Let's try to solve the following problem:

$$\begin{cases} x^3 + y^3 = 8 \\ x^2 + y^2 = 4 \end{cases}.$$

restart;
_EnvExplicit:=true;
sys:=({x^3+y^3=8,x^2+y^2=4});
s1:=solve(sys,{x,y});
map(subs,[s1],sys);

$\_EnvExplicit := true$

$sys := \{x^2 + y^2 = 4, x^3 + y^3 = 8\}$

$s1 := \{x = 0, y = 2\}, \{x = 2, y = 0\}, \{x = -2 - \sqrt{2}\, I, y = -2 + \sqrt{2}\, I\},$
$\{x = -2 + \sqrt{2}\, I, y = -2 - \sqrt{2}\, I\}$

$[\{4 = 4, 8 = 8\}, \{4 = 4, 8 = 8\},$
$\{(-2 - \sqrt{2}\, I)^2 + (-2 + \sqrt{2}\, I)^2 = 4, (-2 - \sqrt{2}\, I)^3 + (-2 + \sqrt{2}\, I)^3 = 8\},$
$\{(-2 - \sqrt{2}\, I)^2 + (-2 + \sqrt{2}\, I)^2 = 4, (-2 - \sqrt{2}\, I)^3 + (-2 + \sqrt{2}\, I)^3 = 8\}]$

Maple has completed and verified the solution, but as we can see, for complex conjugate roots there is no explicit expression for the correctness of the solution. We correct the verification of the solution and connect the *simplify* transformation command [3]: **simplify(map(subs,[s1],sys));**

$[\{4 = 4, 8 = 8\}, \{4 = 4, 8 = 8\}, \{4 = 4, 8 = 8\}, \{4 = 4, 8 = 8\}]$

Now the solution has been verified with explicit derivation of expressions. As you can see, the Maple system finds solutions to systems of algebraic equations, the left side of which is symmetric polynomials.

Let us consider the solution of a system with asymmetric equations, when solving which an auxiliary parameter was introduced:

$$\begin{cases} x^3 - y^3 = 5 \\ xy^2 - x^2y = 1 \end{cases}.$$

We enter the commands of the program for solving a system of algebraic equations:

restart;
_EnvExplicit:=true;
sys:=({x^3-y^3=5,x*y^2-x^2*y=1});
s1:=solve(sys,{x,y});
simplify(map(subs,[s1],sys));

$\_EnvExplicit := true$

$$sys := \{-x^2 y + x y^2 = 1, x^3 - y^3 = 5\}$$

$$s1 := \{x = 1 + \frac{\sqrt{2}}{2}, y = -1 + \frac{\sqrt{2}}{2}\}, \{x = 1 - \frac{\sqrt{2}}{2}, y = -1 - \frac{\sqrt{2}}{2}\}, \Bigg\{x = -\frac{16\left(\frac{1}{2}+\frac{1}{4}I\sqrt{6}+\frac{\sqrt{-10+4I\sqrt{6}}}{4}\right)^3}{7} + 4\left(\frac{1}{2}+\frac{1}{4}I\sqrt{6}+\frac{\sqrt{-10+4I\sqrt{6}}}{4}\right)^2 - \frac{45}{14} - \frac{7}{4}I\sqrt{6} - \frac{7\sqrt{-10+4I\sqrt{6}}}{4}, y = \frac{1}{2} + \frac{1}{4}I\sqrt{6} + \frac{\sqrt{-10+4I\sqrt{6}}}{4}\Bigg\},$$

$$\Bigg\{x = -\frac{16\left(\frac{1}{2}+\frac{1}{4}I\sqrt{6}-\frac{\sqrt{-10+4I\sqrt{6}}}{4}\right)^3}{7} + 4\left(\frac{1}{2}+\frac{1}{4}I\sqrt{6}-\frac{\sqrt{-10+4I\sqrt{6}}}{4}\right)^2 - \frac{45}{14} - \frac{7}{4}I\sqrt{6} + \frac{7\sqrt{-10+4I\sqrt{6}}}{4}, y = \frac{1}{2} + \frac{1}{4}I\sqrt{6} - \frac{\sqrt{-10+4I\sqrt{6}}}{4}\Bigg\},$$

$$\Bigg\{x = -\frac{16\left(\frac{1}{2}-\frac{1}{4}I\sqrt{6}+\frac{\sqrt{-10-4I\sqrt{6}}}{4}\right)^3}{7} + 4\left(\frac{1}{2}-\frac{1}{4}I\sqrt{6}+\frac{\sqrt{-10-4I\sqrt{6}}}{4}\right)^2 - \frac{45}{14} + \frac{7}{4}I\sqrt{6} - \frac{7\sqrt{-10-4I\sqrt{6}}}{4}, y = \frac{1}{2} - \frac{1}{4}I\sqrt{6} + \frac{\sqrt{-10-4I\sqrt{6}}}{4}\Bigg\},$$

$$\Bigg\{x = -\frac{16\left(\frac{1}{2}-\frac{1}{4}I\sqrt{6}-\frac{\sqrt{-10-4I\sqrt{6}}}{4}\right)^3}{7} + 4\left(\frac{1}{2}-\frac{1}{4}I\sqrt{6}-\frac{\sqrt{-10-4I\sqrt{6}}}{4}\right)^2 - \frac{45}{14} + \frac{7}{4}I\sqrt{6} + \frac{7\sqrt{-10-4I\sqrt{6}}}{4}, y = \frac{1}{2} - \frac{1}{4}I\sqrt{6} - \frac{\sqrt{-10-4I\sqrt{6}}}{4}\Bigg\}$$

$$[\{1=1, 5=5\}, \{1=1, 5=5\}, \{1=1, 5=5\}, \{1=1, 5=5\}, \{1=1, 5=5\}, \{1=1, 5=5\}]$$

Discussion of scientific results. All solutions of the system are complex conjugate numbers. The program for solving systems of algebraic equations with symmetric polynomials in the Maple package solves systems with an asymmetric right-hand side. Thus, solving systems of equations in the Maple package allows you to speed up and simplify the execution of complex actions, calculations and eliminate the occurrence of errors. But at the same time, in order for the program to be implemented, you need to correctly apply commands and formulate the task correctly.

Conclusion. A mathematical program for solving systems of higher degree equations will allow you to find correct solutions while reducing computation time.

The article was able to show the development of algorithms and the creation of a mathematical program for finding an analytical solution to linear differential equations with constant coefficients in the Maple environment.

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MAPLE КОМПЬЮТЕРЛІК МАТЕМАТИКА ЖҮЙЕСІНДЕГІ СИМЕТРИЯЛЫҚ ПОЛИНОМАЛДАРДЫ ЕСЕПТЕУ

Аңдатпа. Мақалада симметриялы көпмүшелердің теориясы зерттелген. Зерттеу нәтижесінде жоғары дәрежелі теңдеулер жүйесін шешу әдістерінің бірі симметриялы көпмүшеліктерді қолдану әдісі екендігі көрсетілді. Есептерді шешу әдісінің практикалық көрінісі Maple компьютерлік математикалық жүйесінде жүзеге асырылды.

Тірек сөздер: симметриялық көпмүшеліктер, жоғары дәрежелі теңдеулер, альтернативті шешу әдістері, аналитикалық есептеулер.

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ВЫЧИСЛЕНИЕ СИММЕТРИЧНЫХ МНОГОЧЛЕНОВ В СИСТЕМЕ КОМПЬЮТЕРНОЙ МАТЕМАТИКИ MAPLE

Аннотация. В статье изучается теория симметричных полиномов. В результате исследования было показано, что одним из методов решения систем уравнений высшей степени является метод с использованием симметричных многочленов. Практическая реализация метода решения имеет компьютерную реализацию в системе Maple.

Ключевые слова: симметричные полиномы, уравнения высших степеней, альтернативные методы решения, аналитические вычисления.