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USING CLASSICAL ANALYSIS AND COMPUTER TECHNOLOGIES TO STUDY POWER-EXPONENTIAL TYPE EQUATIONS

Abstract. In the present paper basing on properties of the power and the exponential functions we study power-exponential type equations and establish properties of their roots. Since such equations are transcendental there is no method to find exact roots. But using well known methods of classical calculus and tools of the system of analytical calculations Maple we can establish the boundaries of roots and predict their convergence as the degree tends to infinity. We also give qualitative and quantitative analysis of the difference between the power and the exponential functions for large values of the argument and suggest a convenient formula for approximate but fast calculations. Results of calculations demonstrated in tables and graphs were obtained using Maple program.

Keywords: Maple system, transcendental equation, roots, power function, exponential function.

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Introduction. It is well known that for any real $a>1$ and $\beta>0$ the exponential function a^x will grow much faster than the power function x^β, starting from some (possibly very large) positive value of the variable x. Such a conclusion follows from the value of the limit $\lim_{x\to+\infty} a^x x^{-\beta}=+\infty$. However, this fact does not contain any information about exactly from what values of x the inequality $a^x x^{-\beta}>1$ will be established. We are only sure that this will happen for large values of x. Here we consider the special case $a=\beta=n$ only. Obviously, at the point $x=0$ the inequality $1=n^x>x^n=0$ holds for all natural n. The case $n=1$ is also obvious. Therefore throughout the text we consider power-exponential type equation $n^x=x^n$ assuming $x\in R\setminus\{0\}$ and $n\in N\setminus\{1\}$. The main results are contained in the following three theorems.

Theorem 1. The following assertions are hold for $n\geq 2$*:*

(1) If $n=2$ then $2^x=x^2$ has exactly three distinct roots $x_1<0, x_2=2$ and $x_3=4$. In addition, $2^x<x^2$ for $x\in(-\infty,x_1)\cup(2,4)$ and $2^x>x^2$ for $x\in(x_1,2)\cup(4,+\infty)$.

(2) If n is even and $n \geq 4$ then $n^x = x^n$ has exactly three distinct roots $x_1 < x_2 < x_3$ such that $x_1 < 0, x_2 > 0$ and $x_3 = n$. Moreover, $n^x < x^n$ for $x \in (-\infty, x_1) \cup (x_2, n)$ and $n^x > x^n$ for $x \in (x_1, x_2) \cup (n, +\infty)$ (Fig. 1).

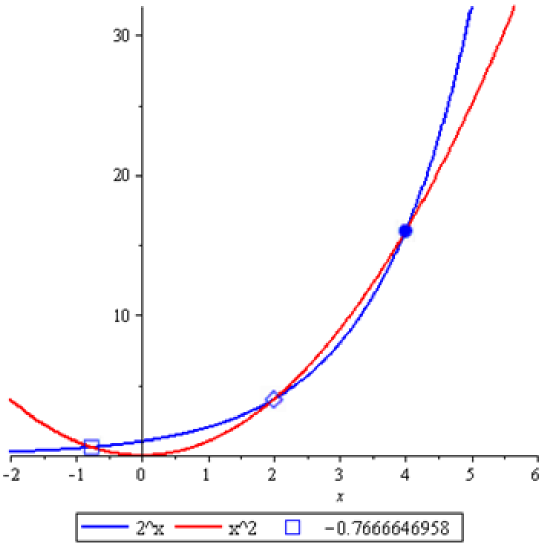

Fig. 1. The graphs of n^{x} and x^{n} at $\mathrm{n} = 2$

(3) If n is odd and $n \geq 3$ then $n^x = x^n$ admits only two positive roots $x_2 > 0$, $x_3 = n$ such that $x_2 < x_3$ and $n^x < x^n$ for $x \in (x_2, n)$ and $n^x > x^n$ for $x \in (-\infty, x_2) \cup (n, +\infty)$ (Fig. 2).

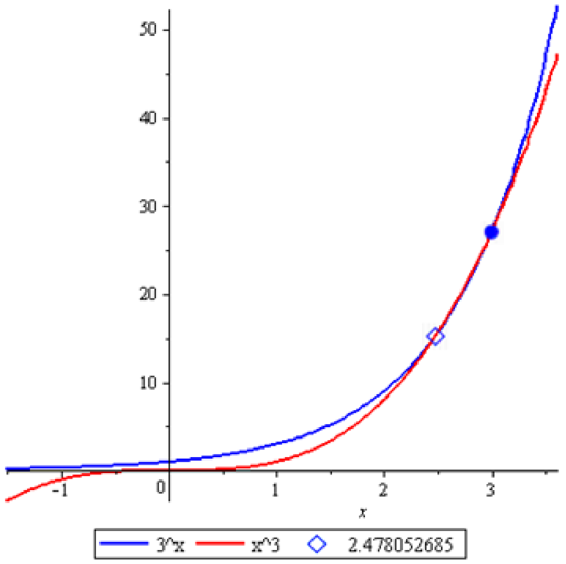

Fig. 2. The graps of n^x and x^n at $n = 3$

Theorem 2. The following estimations hold for the roots $x_1 = x_1(n) < 0$ and $x_2 = x_2(n) > 0$ of the equation $n^x = x^n$ independently of values of *n:*

$$-1 < x_1(n) < -e^{-1}, \qquad 1 < x_2(n) < e \tag{1}$$

where, e is the base of the natural logarithms with an approximative value $e \approx 2.71\ldots$. The roots $x_1(n)$ and $x_2(n)$ both decrease for large n and have the following limits as n tends to infinity:

$$\lim_{n \to +\infty} x_1(n) = -1, \quad \lim_{n \to +\infty} x_2(n) = 1. \tag{2}$$

In Table 1 approximative values of the roots $x_1(n)$ and $x_2(n)$ are shown for some values of *n*.

Table 1
The roots $x_1(n)$, $x_2(n)$ and $x_3(n)$ of the equation $n^x = x^n$ for increasing *n*.

n	$x_1(n)$	$x_2(n)$	$x_3(n)$
2	-0.7667	2	4
3		2.4781	3
4	-0.7667	2	4
5		1.7649	5
10	-0.8267	1.3713	10
50	-0.9298	1.0889	50
100	-0.9569	1.0495	100
150	-0.9682	1.0352	150

Theorem 3. The following inequality holds for all $x \geq n + 1 \geq 4$:

$$n^x - x^n > n^n(n - e) \tag{3}$$

with a consequence $n^x - x^n > n^n(n - 3)$ for large *n*.

Auxiliary results. Introduce the function $F(x) = lnf(x)$, where $f(x) = n^x x^{-n}, x \neq 0$. Such an idea follows from the formula $F'(x) = \frac{f'(x)}{f(x)}$ known in classical analysis. Note that $F(x)$ is defined for all $x \neq 0$ for even n and only for $x > 0$ if n is odd. $F(x)$ is not defined if $x < 0$ and n is odd, but there is no necessity introduce something in this case since $n^x > x^n$ is satisfied automatically: $n^x > 0 > x^n$ for all $x < 0$. Note also that the signs of the derivatives f' and F' coincide on the domain of F, because $f > 0$. Therefore f and F admit the same intervals of increasing and decreasing. In addition,

$$F(x) = 0 \ \leftrightarrow \ f(x) = 1 \ \leftrightarrow n^x = x^n$$

According to definition of F. It is clear also that $F > 0$ is equivalent to $f > 1$ that means $0 < x^n < n^x$, analogously, $F < 0$ is equivalent to $0 < f < 1$ that means $n^x < x^n$.

Lemma 1. For n ≥ 2 the following assertions are true:

1) If n is even then equation $F(x) = 0$ admits one negative root x_1 and two positive roots $x_2 < x_3$ such that $F(x) < 0$ for $x \in (-\infty, x_1) \cup (x_2, x_3)$ and $F(x) > 0$ for $x \in (x_1, 0) \cup (0, x_2) \cup (x_3, +\infty)$.

2) If n is odd then $F(x) = 0$ admits only two positive roots $x_2 < x_3$ such that $F(x) < 0$ for $x \in (x_2, x_3)$ and $F(x) > 0$ for $x \in (0, x_2) \cup (x_3, +\infty)$.

In addition, $x_2 = 2, x_3 = 4$ if $n = 2$ and $x_3 = n$ if $n > 2$ (Fig. 3).

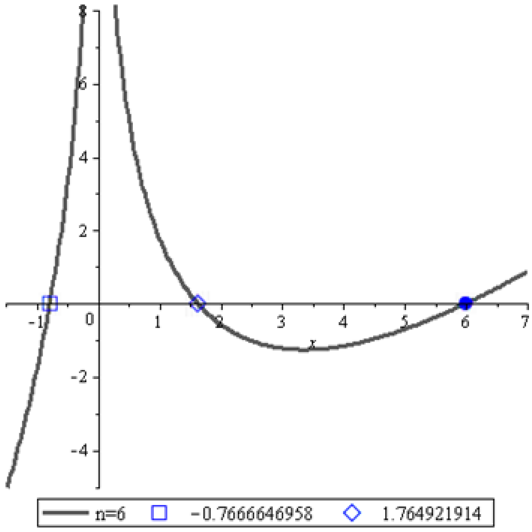

Fig. 3. The graph of $F(x) = lnf(x)$, where $f(x) = n^x x^{-n}$ at $n = 6$.

Proof. 1) For all $n \geq 2$ the function $F(x) = lnf(x)$ can be rewritten in the following form:

$$F(x) = \begin{cases} xln\, n - n\ln|x|, & x \neq 0, \quad \text{if } n \text{ is even,} \\ xln\, n - n\ln x, & x > 0, \quad \text{if } n \text{ is odd.} \end{cases}$$

Its derivative is

$$F'(x) = \begin{cases} ln\, n - nx^{-1}, & x \neq 0, \quad \text{if } n \text{ is even,} \\ ln\, n - nx^{-1}, & x > 0, \quad \text{if } n \text{ is odd.} \end{cases}$$

The case A. Let n be even and $x < 0$. Then $F'(x) > 0$ for all $x < 0$. This implies that $F(x)$ is increasing strictly on the interval $x < 0$. Hence F have to intersect the coordinate line $y = 0$ on some unique point $x_1 \in (-\infty, 0)$ from down to up, in other words, $F(x_1) = 0$. Moreover $F(x) < 0$ at $x < x_1$ and $F(x) > 0$ at $x_1 < x < 0$.

The case B. Let n be any and $x > 0$. The equation $F'(x) = 0$ has an unique root $x^* = n(\ln n)^{-1}$. Since the second derivative $F''(x) = nx^{-2}$ is positive for all $x \neq 0$ at the point x^* the function F attains its least value

$$F(x^*) = \inf_{x>0} F(x) = n - nln\left(\frac{n}{\ln n}\right) = n\,\left(1 - \ln\left(\frac{n}{\ln n}\right)\right). \tag{4}$$

It is easy to show that for $x > 1$ the function $x(\ln x)^{-1}$ satisfies the inequality

$$x(\ln x)^{-1} \geq e,$$

Reaching the equality at $x = e$ only (Fig. 4). Hence $n(\ln n)^{-1} > e$ for all $n > 1$. This implies that $F(x^*) < 0$. From $F''(x) > 0$ and $F(x^*) < 0$ it follows then $F(x)$ has exactly two positive roots x_2 and x_3 such that $x_2 < x^* < x_3$, moreover, $F(x) < 0$ at $x \in (x_2, x_3)$ and $F(x) > 0$ at $x \in (0, x_2) \cup (x_3, +\infty)$.

Uniting now the results of the cases A and B we get the proofs of the assertions in 1) and 2).

To prove the additional assertion observe that $x = n$ is a root of $F(x) = 0$. This means that either x_2 or x_3 coincides with *n*. At $n = 2$ the inequality $x^* > n$ implies $x_2 = 2$. Then $x_3 = 4$. At $n \geq 3$ we have $x^* < n$. Then $x_3 = n$ in order to satisfy $x_2 < x^* < x_3$. Lemma 1 is proved.

Proof of Theorem 1. (1) Assume that $n = 2$. Then clearly, $F(x) < 0$ on the interval $(-\infty, x_1)$, where $x_1 < 0$. This means that $0 < f(x) = n^x x^{-n} < 1$, or equivalently $n^x < x^n$. Moreover, $F(x)$ increases on $(-\infty, x_1)$, therefore $f(x)$ increases too implying that the value of $n^x x^{-n}$ increases and tends to 1 (n^x overtakes x^n) as *x* tends to x_1 from the left.

On the interval $(x_1, 0)$ the function $F(x)$ increases and satisfies $F(x) > 0$. Therefore $f(x) > 1$ on $(x_1, 0)$ meaning $n^x > x^n$ (with growing $n^x x^{-n}$). At $x = 0$ the inequality $n^x > x^n$ is preserved. On the interval (0,2) we still have $F(x) > 0$ meaning $n^x > x^n$. However the fraction $n^x x^{-n}$ decreases and tends to 1 (x^n overtakes n^x) as x tends to 2 from the left, because decreasing of $F(x)$. On (2,4) we have $F(x) < 0$ which means $0 < f(x) < 1$, equivalently $n^x < x^n$. The behavior of $n^x x^{-n}$ near $x = 2$ and $x = 4$ can be analyzed by the same way. In final, clearly that $n^x > x^n$ for all $x > 4$ due to $F(x) > 0$. Moreover, $n^x x^{-n}$ increases, because so does $F(x)$.

(2) n is even and $n \geq 4$. The proof will be the same as in the previous case by replacing only 2 to x_2 and 4 to *n*.

(3) *n* is odd and $n \geq 3$. The proof is clear now. Theorem 1 is proved.

Proof of Theorem 2. In the proof of Theorem 1 we obtained in fact rough estimates $x_1 < 0$ and $0 < x_2 < n(\ln n)^{-1}$ for roots. Here we will improve them. Since F can admit a negative root only if n is even, using the corresponding expression $F(x) = x\ln n - n\ln|x|$ we find

$$F(-1) = -\ln n \leq -\ln 2 < 0, \qquad F(-e^{-1}) = -e^{-1}\ln n + n > 0,$$

Where the second inequality follows from formula (4). The continuous function $F(x) = x\ln n - n\ln|x|$ changes its signs at the endpoints of the segment $[-1, -e^{-1}]$. Therefore the unique negative root x_1 is still in $(-1, -e^{-1})$ proving the first inequality in (1). The second inequality in (1) follows from $F(1) = \ln n > 0$ and $F(e) = e\ln n - n < 0$.

Let us seek for x_2 as a least of two positive roots of $F(x) = x\ln n - n\ln x = 0$ equivalent to $\frac{x}{\ln x} = \frac{n}{\ln n}$. Denoting $C(n) := \frac{n}{\ln n}$ we obtain the equation

$$x(\ln x)^{-1} = C(n),$$

Depending on the parameter $C(n)$. Since $lim_{n\to\infty} C(n) = +\infty$ and $x(\ln x)^{-1}$ decreases for $1 < x < e$ and admits a vertical asymptote $x = 1$ due to $\lim_{x\to 1+} x(\ln x)^{-1} = +\infty$, then the value of the function $x(\ln x)^{-1}$ can be equal to $C(n)$ only at an unique point (denote it x_2) which depends on n and converges to 1 from the right as $n \to \infty$ (Fig. 4). The second equality in (2) is proved.

By the same say we can predict the behavior of the negative root x_1 of $n^x = x^n$ which occurs at even n only. Since $n^x = x^n$ is equivalent to $F(x) =$

$x\ln n - n\ln|x| = 0$ on the interval $(-\infty, 0)$ we get $\frac{x}{\ln(-x)} = \frac{n}{\ln n}$, where introducing a new variable $x = -t$ yields a new equation

$$t(\ln t)^{-1} = -C(n).$$

We know that $t(\ln t)^{-1}$ decreases on the interval $(0,1)$ and has limits $\lim_{t\to 1-0} t(\ln t)^{-1} = -\infty$ and $\lim_{t\to +0} t(\ln t)^{-1} = 0$. Therefore $t(\ln t)^{-1} = -C(n)$ admits an unique solution (denote it $t_1(n)$) which belong to $(0,1)$ and converges to 1 from the left as $n \to \infty$, in other words $\lim_{n\to\infty} t_1(n) = 1$.

Pass to the old variable $x = -t$. Then $t_1(n)$ corresponds to $x_1(n) = -t_1(n)$, which converges to -1 from the right as $n \to \infty$. The first equality in (2) is proved. Theorem 2 is proved.

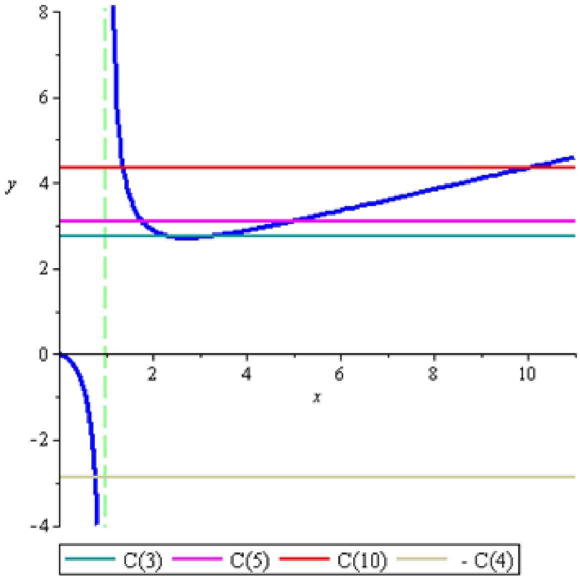

Fig. 4. The graph of the function $y = x(\ln x)^{-1}$ and straight lines $y = C(n)$ at $n = 3, 5, 10$ and $y = -C(4)$.

Proof of Theorem 3. Let $n \geq 3$. Introduce the function $\phi(x) = n^x - x^n$. Since $\ln(1+t) < t$ at $t > 0$ then for all $x > n$ we have $x^n = n^n e^{n\ln\left(1+\frac{x-n}{n}\right)} < n^n e^{x-n}$. This yields

$$\phi(x) = n^x - x^n > n^n(n^{x-n} - e^{x-n}) \quad (5)$$

For all $x > n$. In particular (5) implies $\phi(n+1) > n^n(n-e)$ at $x = n+1$. Since $\phi(x)$ increases for $x > n$ the inequality $\phi(x) \geq \phi(n+1)$ is preserved for all $x \geq n+1$. The inequality (3) is proved. The inequality $n^x - x^n > n^n(n-3)$ follows obviously. Theorem 3 is proved.

In Table 2 results of calculations are shown which confirm Theorem 3. For large n the difference $n^x - x^n$ increases very fast starting from values $x = n+1$. Using formula $n^x - x^n > n^n(n-3)$ we can easily predict that $10^{11} - 11^{10}$ is greater than $7 \cdot 10^{10}$. The exact value $10^{11} - 11^{10}$ of calculated in Maple is 74062575399. Using formula (3) gives 72817181720. Another example is $20^{21} -$

21^{20}. It is greater than the number $20^{20} \cdot (20-3) = 17 \cdot 2^{20} \cdot 10^{20} = 17 \cdot 1024^2 \cdot 10^{20} = 1.7825792 \cdot 10^{27}$, given in Table 2.

The exact value of $20^{21} - 21^{20}$ is 1818933570553048451362803599 or approximatively $1.8189336 \cdot 10^{27}$.

Table 2

Lower bounds of the difference $n^x - x^n$ at $x \geq n+1$.

n	x	$n^x - x^n$	$n^n(n-e)$	$n^n(n-3)$
3	4	17	7.61	0
4	5	399	328	256
5	6	7849	7130	6250
6	7	162287	153112	139968
7	8	3667649	3526179	3294172
8	9	91171007	88612527	83886080
9	10	2486784401	2433666326	2324522934
10	11	74062575399	72817181720	70000000000
20	21	$1.8189336 \cdot 10^{27}$	$1.812119491 \cdot 10^{27}$	$1.7825792 \cdot 10^{27}$

Problem 1. Is it possible to make sharper the right boundaries of estimations (1)? An idea: try to replace e to a smaller number of the kind γe^{α}, where $0 < \alpha \leq 1$ and $0 < \gamma \leq 1$.

Problem 2. We studied mutual behavior of the functions a^x and x^{β} in partial cases $\alpha = \beta = n \in N$. The general case $a \in (0,1) \cup (1, +\infty)$ and $\beta \in R \setminus \{0\}$ requires supplementary studies.

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ДӘРЕЖЕЛІ-КӨРСЕТКІШТІ ТИПТЕГІ ТЕҢДЕУЛЕРДІ ЗЕРТТЕУДЕ КЛАССИКАЛЫҚ ТАЛДАУ МЕН КОМПЬЮТЕРЛІК ТЕХНОЛОГИЯЛАРДЫ ПАЙДАЛАНУ

Аңдатпа. Бұл мақалада біз дәрежелі және көрсеткішті функциялардың қасиеттеріне сүйене отырып, дәрежелі-көрсеткішті типтегі теңдеулерді зерттейміз және олардың тамырларының қасиеттерін көрсетеміз. Мұндай теңдеулер трансцендентті болғандықтан, түбірлердің тура мәндерін табудың ешқандай әдісі болмайды. Дегенмен, классикалық талдаудың белгілі әдістері мен Maple аналитикалық есептеу жүйесінің қаражаттарын қолдану арқылы біз түбірлердің шекараларын көрсете аламыз және дәреже көрсеткіші шексіздікке ұмтылған кезіндегі олардың жинақтылығын алдын ала болжай аламыз. Сонымен бірге, біз дәрежелі және көрсеткішті функциялардың аргументтің үлкен мәндеріндегі айырымының сапалық және сандық талдауын береміз және жылдам жуықтап есептеудің қолайлы формуласын ұсынамыз. Кестелерде келтірілген есептеу нәтижелері мен графиктер Maple программасы көмегімен алынған.

Тірек сөздер: Maple жүйесі, трансценденттік теңдеу, түбір, дәрежелі функция, көрсеткішті функция.

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ИСПОЛЬЗОВАНИЕ КЛАССИЧЕСКОГО АНАЛИЗА И КОМПЬЮТЕРНЫХ ТЕХНОЛОГИЙ В ИЗУЧЕНИИ УРАВНЕНИЙ СТЕПЕННО-ПОКАЗАТЕЛЬНОГО ТИПА

Аннотация. В данной работе опираясь на свойства степенной и показательной функций, мы исследуем степенно-показательные уравнения, устанавливаем свойства их корней. Так как такие уравнения являются трансцендентными, нет метода нахождения точных значений корней. Тем не менее, используя известные методы классического анализа и средства системы аналитических вычислений Maple, мы можем определить границы корней и предсказывать их сходимость при стремлении степеней к бесконечности. Мы также даем качественный и количественный анализ разности между степенной и показательной функциями при больших значениях аргумента и предлагаем удобный метод приближенного, но быстрого счета. Результаты вычислений, приведенные в таблицах, и графики получены с помощью Maple программы.

Ключевые слова: система Maple, трансцендентное уравнение, корень, степенная функция, показательная функция.