IRSTI 27.47.17

Yu.R. Krahmaleva ©

PhD

https://orcid.org/0000-0003-2166-3885

M.Kh. Dulaty Taraz Regional University,

Taraz, Kazakhstan

yuna_kr@mail.ru

https://doi.org/10.55956/RZBK3395

BESSEL EQUATION AND FUNCTIONS IN COMPUTER MATHEMATICS SYSTEM

Abstract. The relevance of technical issues of solving problems of mathematical physics is growing every year. The development of computer technologies leads to the use of modern methods for their solution. The application of analytical computing systems is considered as an effective method, which contributes to their productive implementation. The article deals with finding a general solution of the Bessel equation and equations leading to it in the Maple program, graphs of functions are plotted.

Keywords: special point, Bessel function of the 1st kind, Bessel equation, Bessel function of the 2nd kind, asymptotic behaviour.

Krahmaleva Yu.R. Bessel equation and functions in computer mathematics system // Mechanics and Technologies / Scientific journal. – 2024. – No.2(84). – P.426-439. https://doi.org/10.55956/RZBK3395

Introduction. When solving various problems of mathematical physics we deal with so-called special functions The term "special functions" means all mathematical non-elementary functions. Along with special functions the equivalent term "higher transcendental functions" is used. The class of these functions has a characteristic difference from many other functions, which manifests itself when solving equations of the form [1]:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]-q(x)y=0 .$$

The equations have special points. The coefficient $p(x)$ takes zero value: $p(x)=0$ at possibly one or more points in the interval of change of the variable . In these equations special functions act as its solution. Having a certain specificity, the scope of application of special functions in a variety of applied problems expands. One of the representatives of special functions are cylindrical functions or Bessel functions[1,2].

Materials and methods. Let us consider the Bessel equation:

$$x^2y'' + xy' + \left(x^2 - v^2\right)y = 0, \tag{1}$$

which is a homogeneous linear differential equation with variable coefficients. The order (1) is determined by the constant v. Equation (1) owes its name to the German mathematician Friedrich Wilhelm Bessel. The scientist conducted research in 1824. The result led to the discovery of a special class of functions, which were called Bessel functions or cylindrical functions. These functions and represent the solution of the equation. [3]

To find the general solution, a technique is used. The equation is multiplied by the multiplier $\frac{1}{x^2}$:

$$y'' + \frac{1}{x}y' + \frac{x^2 - v^2}{x^2}y = 0. \tag{2}$$

Simultaneously with the equation (2), we consider an equation of the form:

$$y'' + p(x)y' + q(x)y = 0. \tag{3}$$

assuming the existence of a special point of the equation. For definiteness, suppose $x = 0$. Let the coefficients of equations $p(x)$and $q(x)$be expressed as functional series:

$$p(x) = \sum_{m=0}^{\infty} \frac{a_m x^m}{x}, \quad q(x) = \sum_{m=0}^{\infty} \frac{b_m x^m}{x^2}.$$

The series composed only of numerators in the last functional series converge and have a radius of convergence of $|x| < R$. The following condition is imposed on the coefficients a_m and b_m: $a_0 \neq b_0 \neq b_1 \neq 0$. Then equation (1) has at least one solution, which has a representation in the form of a series:

$$y(x) = x^p \sum_{m=0}^{\infty} c_m x^m. \tag{4}$$

In (4) $c_0 \neq 0$. The series (4) is convergent with a radius of convergence $|x| < R$, To calculate the coefficients of c_k, the expression $y(x)$as (4) is substituted into the original equation (1). Using the method of indeterminate coefficients, determine each c_k [3-5].

To calculate the coefficient c_k, a so-called defining equation of the form

$$p(p-1) + a_0 p + b_0 = 0. \tag{5}$$

The coefficients a_0 and b_0 are calculated using the formulas::

$$a_0 = \lim_{x \to 0} xp(x) \tag{6}$$

$$b_0 = \lim_{x \to 0} x^2 q(x) \tag{7}$$

Then for the equation (2) we have:

$$a_0 = \lim_{x \to 0} x \cdot \frac{1}{x} = 1,$$

$$b_0 = \lim_{x \to 0} x^2 \frac{x^2 - v^2}{x^2} = -v^2.$$

Substituting the values of a_0 and b_0 into equation (3):

$$p(p-1) + p - v^2 = 0,$$

whence it follows:

$$p_{1,2} = \pm v.$$

Find the partial solution $y_1(x)$ at $p_1 = v$, using the decomposition (4):

$$y_1(x) = x^p \sum_{m=0}^{\infty} c_m x^m = x^v \sum_{m=0}^{\infty} c_m x^m,$$

substitute the expression $y_1(x)$ в (3) and get the expression [1]:

$$J_v(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \cdot \Gamma(m+v+1)} \left(\frac{x}{2}\right)^{2m+v}, \tag{8}$$

which represents a Bessel function of genus 1 with positive v.
The solution $y_2(x)$ at $p_2 = -v$ has the form :

$$y_2(x) = x^p \sum_{m=0}^{\infty} c_m x^m = x^{-v} \sum_{m=0}^{\infty} c_m x^m.$$

Substituting $y_2(x)$ into (3) , we obtain the second partial solution in the form [1]:

$$J_{-v}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \cdot \Gamma(m-v+1)} \left(\frac{x}{2}\right)^{2m-v}, \tag{9}$$

where $J_{-\nu}(x)$ is a Bessel function of the 1-st genus with negative ν .The general solution of equation $J_{-\nu}(x)$ at non-integer value of ν is written as:

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x), \quad (10)$$

where C_1, C_2 are arbitrary constants $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent functions.

At ν-total: $\nu = n$ the linear independence of functions $J_n(x)$ and $J_{-n}(x)$ is violated, since the functions have a relationship, which is expressed by the formula

$$J_{-n}(x) = (-1)^n J_n(x) \quad (11)$$

To solve the question, we introduce a Bessel function of genus 2 [1]:

$$Y_\nu(x) = \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu}, \quad (12)$$

and the general solution of the equation for any ν will be described by the formula:

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x). \quad (13)$$

where C_1, C_2 is an arbitrary constant, $J_\nu(x)$ and $Y_\nu(x)$ are linearly independent functions. Linear combinations of $J_\nu(x)$ and $Y_\nu(x)$ have the simplest asymptotic expansions at large x and are often encountered in applications [5].

Research results and discussion. Let's consider solving the equation in the Maple program. Enter the equation and the command to solve it [6-9]:

$$eq1 := x^2 \cdot diff(y(x), x\$2) + x \cdot diff(y(x), x) + (x^2 - v^2) \cdot y(x) = 0;$$
$$req1 := dsolve(eq1, y(x));$$

$$eq1 := x^2 \left(\frac{d^2}{dx^2} y(x) \right) + x \left(\frac{d}{dx} y(x) \right) + \left(-v^2 + x^2 \right) y(x) = 0$$
$$req1 := y(x) = \_C1 \, \mathrm{BesselJ}(v, x) + \_C2 \, \mathrm{BesselY}(v, x)$$

The general solution corresponds to the formula (13). Let us substitute $\nu = \frac{3}{4}$ в *eq1*:

$$eq1v1 := subs(v = v1, eq1); \; req11 := dsolve(eq1v1, y(x));$$
$$eq1v1 := x^2 \left(\frac{d^2}{dx^2} y(x) \right) + x \left(\frac{d}{dx} y(x) \right) + \left(x^2 - \frac{9}{16} \right) y(x) = 0$$
$$req11 := y(x) = \_C1 \, \mathrm{BesselJ}\left(\frac{3}{4}, x \right) + \_C2 \, \mathrm{BesselY}\left(\frac{3}{4}, x \right)$$

The general solution corresponds to the formula (13). Let's substitute the integer value $\nu = 2$:

$v2 := 2;\ eq1v2 := subs(v = v2, eq1);\ req1v2 := dsolve(eq1v2, y(x));$

$v2 := 2$

$eq1v2 := x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + (x^2 - 4)\,y(x) = 0$

$req1v2 := y(x) = \_C1\ \mathrm{BesselJ}(2, x) + \_C2\ \mathrm{BesselY}(2, x)$

As can be seen, at different values, program v writes a general solution corresponding to formula (13). Let us consider the examples.

Example 1. Find the general solution of the equation:

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{9x^2}\right)y = 0 \quad (14)$$

Solution. Let's reduce the initial equation to the form (2):

$$y'' + \frac{1}{x}y' + \frac{9x^2 - 1}{9x^2}y = 0,$$

$$y'' + \frac{1}{x}y' + \frac{9\left(x^2 - \frac{1}{9}\right)}{9x^2}y = 0,$$

$$y'' + \frac{1}{x}y' + \left(\frac{x^2 - \frac{1}{3^2}}{x^2}\right)y = 0.$$

From the last equation we determine v: $v = \frac{1}{3}$. The general solution is written in the form (13):

$$y(x) = C_1 J_{\frac{1}{3}}(x) + C_2 Y_{\frac{1}{3}}(x).$$

Let's find the general solution of the equation (14) in the Maple program. [6-9] We have:

$eq2 := diff(y(x), x\$2) + \frac{1}{x} \cdot diff(y(x), x) + \left(1 - \frac{1}{9 \cdot x^2}\right) \cdot y(x) = 0;$

$req2 := dsolve(eq2, y(x));$

$eq2 := \frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{x} + \left(1 - \frac{1}{9x^2}\right)y(x) = 0$

$req2 := y(x) = \_C1\ \mathrm{BesselJ}\left(\frac{1}{3}, x\right) + \_C2\ \mathrm{BesselY}\left(\frac{1}{3}, x\right)$

As can be seen, when the equation (14) was introduced into the program, no transformations were required to reduce it to form (2).

Example 2. Find the general solution of the equation:

$$x^2y'' + xy' + 4\cdot\left(x^4 - 2\right)y = 0 \quad (15)$$

Solution. To fully match the form of equation (1), we insert a new variable z : $z = x^2$. Let's write the equation through z, for this purpose let's calculate y' and y'' elements of the equation. We have:

$$x = \sqrt{z}, \quad y' = \frac{dy}{dx} = \frac{dz}{dx}\cdot\frac{dy}{dz} = 2x\frac{dy}{dz} = 2\sqrt{z}\frac{dy}{dz};$$

$$y'' = \left(2\sqrt{z}\frac{dy}{dz}\right)' = \sqrt{z}\frac{d}{dz}\left(\sqrt{z}\frac{dy}{dz}\right) = 2\frac{dy}{dz} + 4z\frac{d^2y}{dz^2}.$$

After substitution y', y'', $x = \sqrt{z}$ and $x^2 = z$ into the original equation, it will take the form:

$$z\left(2\frac{dy}{dz} + 4z\frac{d^2y}{dz^2}\right) + 2\sqrt{z}\cdot\sqrt{z}\frac{dy}{dz} + 4\cdot\left(z^2 - 2\right)y = 0,$$

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + \left(z^2 - 2\right)y = 0.$$

The last equation is the Bessel equation with the function $y = y(z)$, in which $v = \sqrt{2}$. The general solution of the equation has the form:

$$y(z) = C_1 J_{\sqrt{2}}(z) + C_2 Y_{\sqrt{2}}(z). \quad (16)$$

Let's pass to the x- variable of the initial equation and write down its general solution:

$$y(x) = C_1 J_{\sqrt{2}}\left(x^2\right) + C_2 Y_{\sqrt{2}}\left(x^2\right). \quad (17)$$

Now let's solve the equation (15) in Maple:

$$eq3 := x^2\cdot diff(y(x), x\$2) + x\cdot diff(y(x), x) + 4\cdot\left(x^4 - 2\right)\cdot y(x) = 0;$$
$$req3 := dsolve(eq3, y(x));$$

$$eq3 := x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + 4\left(x^4 - 2\right)y(x) = 0$$

$$req3 := y(x) = \_C1\ \mathrm{BesselJ}\left(\sqrt{2}, x^2\right) + \_C2\ \mathrm{BesselY}\left(\sqrt{2}, x^2\right)$$

As can be seen, the solution coincides completely with the general solution (17), according to the technique of finding the solution of the Bessel equation.

Let us consider the equations reduced to the Bessel equation (1). An equation of the form:

$$x^2y'' + xy' + \left(k^2x^2 - v^2\right)y = 0, \quad (18)$$

with a constant $k : k \neq 0$ by substitution $\tau = kx$ is reduced to an equation of the form:

$$\tau^2 \frac{d^2 y}{d\tau^2} + \tau \frac{dy}{d\tau} + \left(\tau^2 - v^2\right) y = 0 \quad (19)$$

The equation (19) is a Bessel equation. As in the case of solving the equation (15), we need to perform a computational process with a double jump to the new variable, then return to the old variable . Minimizing the solution time is possible when solving the equation in Maple. For example, we need to find a general solution to the equation:

$$x^2 y'' + xy' + \left(9x^2 - 2\right) y = 0 \quad (20)$$

Equation (20) in structure corresponds to the equation (18). Then we have:

$$eq4 := x^2 \cdot diff(y(x), x\$2) + x \cdot diff(y(x), x) + \left(9 \cdot x^2 - 2\right) \cdot y(x) = 0;$$
$$req4 := dsolve(eq4, y(x))$$

$$eq4 := x^2 \left(\frac{d^2}{dx^2} y(x) \right) + x \left(\frac{d}{dx} y(x) \right) + \left(9x^2 - 2\right) y(x) = 0$$
$$req4 := y(x) = \_C1 \, \mathrm{BesselJ}\left(\sqrt{2}, 3x\right) + \_C2 \, \mathrm{BesselY}\left(\sqrt{2}, 3x\right).$$

To the Bessel equation (1) is given an equation having the form:

$$x^2 y'' + axy' + \left(b + cx^m\right) y = 0, \quad (21)$$

with constants a, b, c, m: $c > 0$, $m \neq 0$. Using substitutions for the variable $x = \left(\frac{z}{\gamma}\right)^{\frac{1}{\beta}}$ and the function $y = \left(\frac{1}{\gamma}\right)^{-\frac{\alpha}{\beta}} \cdot u$ the equation will be composed in the following form:

$$z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} + \left(z^2 - v^2\right) u = 0, \quad (22)$$

where $\alpha = \frac{a-1}{2}$, $\beta = \frac{m}{2}$, $\gamma = \frac{2\sqrt{c}}{m}$, $v^2 = \frac{(a-1)^2 - 4b}{m^2}$.

Let's find the solution of the equation:

$$y'' + \frac{3}{x} y' + 4y = 0 \quad (23)$$

refers to an equation of the form (21). Solving in Maple allows you to obtain a general solution without prior substitutions:

$$eq5 := diff(y(x), x\$2) + \frac{3}{x} \cdot diff(y(x), x) + 4 \cdot y(x) = 0;\ req5 := dsolve(eq5, y(x));$$

$$eq5 := \frac{d^2}{dx^2} y(x) + \frac{3\left(\frac{d}{dx} y(x)\right)}{x} + 4\, y(x) = 0$$

$$req5 := y(x) = \frac{\_C1 \operatorname{BesselJ}(1, 2x)}{x} + \frac{\_C2 \operatorname{BesselY}(1, 2x)}{x}$$

In the Malpe program, the commands used to call Bessel functions are:
BesselJ(v,x)- Bessel function of the 1-st kind;
$\operatorname{BesselJ}(\mathrm{v}, x);$
$\operatorname{BesselJ}(v, x)$

BesselY(v,x)- Bessel function of the 2-nd kind;
$\operatorname{BesselY}(\mathrm{v}, x);$
$\operatorname{BesselY}(v, x)$

The function $J_v(x)$ in the form of a series according to the formula (8) at $m = p$ is inserted in the program as follows [8]:

$$J1 := (v, x) \rightarrow sum((((-1)\hat{}p) / (\operatorname{factorial}(p) * \operatorname{GAMMA}(p + v + 1)) * ((x/2)\hat{}(2*p + v))), p = 0 ..infinity);$$

$$J1 := (v, x) \rightarrow \sum_{p=0}^{\infty} \frac{(-1)^p \left(\frac{1}{2} x\right)^{2p+v}}{p!\, \Gamma(p + v + 1)}$$

Use the plot command to *plot* the graphs of the function $J_v(x)$ for various non-integer values v :

$GR := plot\left(\left[\operatorname{BesselJ}\left(\frac{3}{5}, x\right), \operatorname{BesselJ}\left(\frac{4}{5}, x\right), \operatorname{BesselJ}\left(\frac{6}{5}, x\right), \operatorname{BesselJ}\left(\frac{7}{5}, x\right)\right], x = -8 ..10,\right.$
$color = [red, \mathrm{green, blue, magenta}], thickness = [1, 2, 3, 4], legend = \left[\text{'BesselJ}\left(\frac{3}{5}, x\right)\text{'},\right.$
$\text{'BesselJ}\left(\frac{4}{5}, x\right)\text{'},$
$\left.\text{'BesselJ}\left(\frac{6}{5}, x\right)\text{','BesselJ}\left(\frac{7}{5}, x\right)\text{'}\right], linestyle = [solid, longdash, dashdot, dot],$
$caption =$ " *Graphs of the first four Bessel functions of genus I with* v>0",
$\left. captionfont = [TIMES, ROMAN, 14]\right) :$
$GR;$

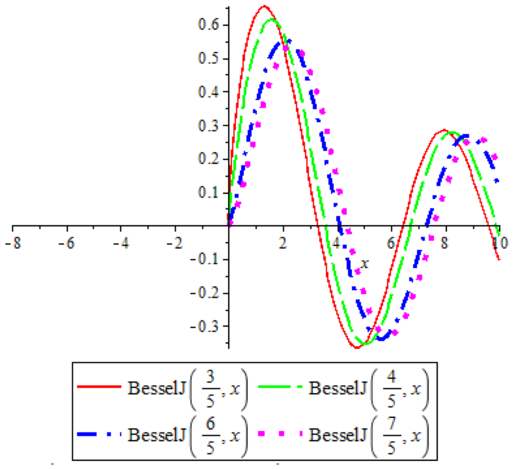

Fig. 1. Graphs of the first four Bessel functions of genus 1 with v > 0

Let's draw the graphs of the functions $J_v(x)$ for a whole v: $v=\overline{0,4}$ have the form :

```
GR := plot([BesselJ(1, x), BesselJ(2, x), BesselJ(3, x), BesselJ(4, x)], x =-8 ..10,
color = [red, green, blue, magenta], thickness = [1, 2, 3, 4], legend = ['BesselJ(1, x)',
    'BesselJ(2, x)',
'BesselJ(3, x)','BesselJ(4, x)'], linestyle = [solid, longdash, dashdot, dot],
caption = "Графики первых четырех функций Бесселя I-го рода с v>0",
captionfont = [TIMES, ROMAN, 14]) :
GR;
```

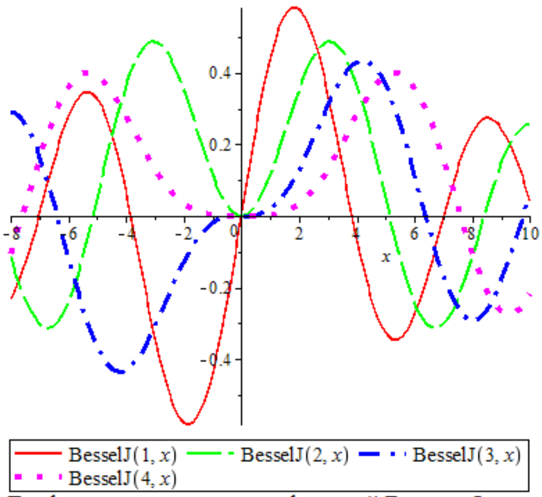

Fig. 2. Graphs of the first four Bessel functions of genus I with v > 0

The Bessel function of 1-st kind $J_v(x)$ at $x \geq 0$, $v \geq 0$ is a real function with argument x. The function is bounded. The graph of the function represents oscillations. At large and small x, the behavior of the function is characterized by the expressions:

$$J_\nu(x) \underset{x\to\infty}{\approx} \sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right);$$

$$J_\nu(x) \underset{x\to 0}{\approx} \frac{x^\nu}{2^\nu\Gamma(\nu+1)}.$$

Function $J_\nu(x)$ has an infinite number of zeros. At $\nu>0$, the point $x=0$ belongs to the set of zeros of the function.

In the Maple program we write $J_\nu(x)$ at $m=p$ using the formula (9):

```
J2 := (v, x) → sum((((-1)^p)/(factorial(p)*GAMMA(p-v + 1))*((x/2)^(2*p-v))), p
    = 0 ..infinity);
```

$$J2 := (v,x) \rightarrow \sum_{p=0}^{\infty} \frac{(-1)^p \left(\frac{1}{2}x\right)^{2p-v}}{p!\,\Gamma(p-v+1)}$$

Graphs of functions $J_\nu(x)$ with various negative non integer values ν set with the *plot* command [8]:

```
GR1 := plot([BesselJ(-3/5, x), BesselJ(-4/5, x), BesselJ(-6/5, x), BesselJ(-7/5, x)], x =
    -8..10,
color = [red, green, blue, magenta], thickness = [1, 2, 3, 4], legend= ['BesselJ(-3/5, x)',
    'BesselJ(-4/5, x)',
'BesselJ(-6/5, x)','BesselJ(-7/5, x)'], linestyle= [solid, longdash, dashdot, dot],
caption = "Ggraphs of the first four Bessel functions of genus I with  v<0",
captionfont = [TIMES, ROMAN, 14]) :
GR1;
```

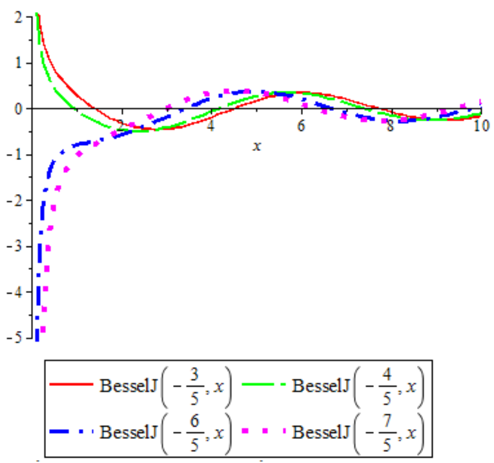

Fig. 3. Graphs of the first four Bessel functions of genus I with v < 0

Graphs of functions $J_v(x)$ for a negative integer v [8]:

```
GRR := plot([BesselJ(-1, x), BesselJ(-2, x), BesselJ(-3, x), BesselJ(-4, x)], x =-8 ..10,
color = [red, green, blue, magenta], thickness = [1, 2, 3, 4], legend = ['BesselJ(-1, x)','BesselJ(
    -2, x)',
'BesselJ(-3 , x)','BesselJ(-4, x)'], linestyle = [solid, longdash, dashdot, dot],
caption = ". Graphs of the first four Bessel functions of genus I  with  v<0",
captionfont = [TIMES, ROMAN, 14]) :
GRR;
```

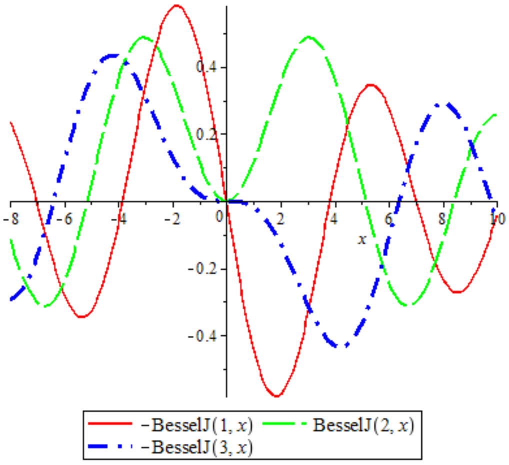

Fig. 4. Graphs of the first four Bessel functions of genus I with v < 0

As shown, the graphs of Bessel functions of the 1-st kind at different values represent damped oscillations. The graph consists of a slowly decreasing part and an oscillating part. Due to this feature, Bessel functions have applications in wave propagation problems.

According to the asymptotic behavior, Bessel functions of the 1-st kind oscillate with the growth of the argument with amplitude $\frac{1}{\sqrt{x}}$.

The expression for $Y_v(x)$ of the Bessel function of genus 2 is given by $J_v(x)$ and $J_{-v}(x)$ and has no explicit representation. But it is possible to graph the function in Maple[8]:

```
GR2 := plot([BesselI(1, x), BesselI(2, x), BesselI(3, x), BesselI(4, x)],
x = 0 ..10, color = [red, green, blue, magenta], thickness = [1, 1, 3, 4], legend = ['BesselI(1, x)',
'BesselI(2, x)','BesselI(3, x)','BesselI(4, x)'], linestyle = [solid],
caption =    "Graphs of the first four Bessel functions of genus II"
captionfont = [TIMES, ROMAN, 14]) :
GR2;
```

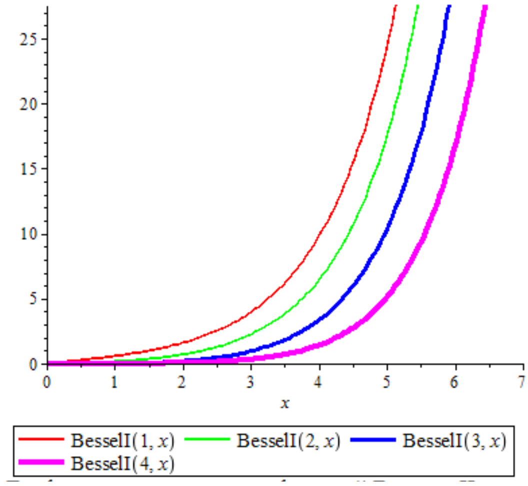

Fig. 5. Graphs of the first four Bessel functions of genus II

The Bessel function of the 2nd kind $Y_v(x)$ is a real function at values $x>0$, $v>0$. The function is bounded at ∞ and with oscillations. Asymptotic formulas characterize $Y_v(x)$ the behavior of the function at large and small values of x:

$$Y_v(x) \underset{x\to\infty}{\approx} \sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{v\pi}{2}-\frac{\pi}{4}\right);$$

$$Y_v(x) \underset{x\to 0}{\approx} -\frac{2^v\Gamma(v)}{\pi x^v}.$$

$$Y_0(x) \underset{x\to 0}{\approx} -\frac{2}{\pi}Ln\frac{2}{x}.$$

For Bessel functions of the 2nd kind, an exponential growth is observed with increasing values of the variable x. The asymptotic behavior of functions at large values of the argument leads to asymptotic formulas that allow us to use convenient representations of functions in the considered domain.

Analytical solution of the Bessel differential equation, as well as equations reduced to it, involves computational operations of introducing new variables to bring to the standard form of equations. Modern computer packages make it possible to avoid routine computational calculations and to find an analytical solution of the equation without introducing additional variables. Given the representation of Bessel functions in the form of a series, it is undoubtedly remarkable to be able to construct functions in Maple, according to which a visual representation of these functions is created, which allows us to study their behavior.

Practical examples of finding the general solution of Bessel's equation and leading to it showed the advantages of computer mathematics programs and as a consequence confirmed the alternative method of finding the solution by means of computer mathematics systems software.

Conclusion. The development of information technologies at the present stage opens new modern methods of solving mathematical problems. The combined method, including methods of the theory of differential equations and realization in the system of computer mathematics, as shown in this article is very productive. The characteristic feature of the method is the efficiency of solution, speed of calculation, which makes it possible to use in solving applied problems.

References

1. Goloskokov D.P. Uravneniya matematicheskoy fiziki. Resheniye zadach v sisteme Maple [Equations of mathematical physics. Solving problems in the Maple system]. – St. Petersburg: Peter, 2004. – 539 p., [in Russian].
2. Bitsadze A.V. Uravneniya matematicheskoy fiziki [Equations of mathematical physics]: Textbook. – 2nd ed., revised and supplemented. – Moscow: Nauka, Main editorial office of physical and mathematical literature, 2012. – 336 p., [in Russian].
3. Koshlyakov N.S. Osnovnyye differentsial'nyye uravneniya matematicheskoy fiziki [Basic differential equations of mathematical physics]. – Moscow: State Publishing House of Physical and Mathematical Literature, 2002. – 276 p., [in Russian].
4. Mikhlin S.G. Kurs matematicheskoy fiziki [Course of mathematical physics]. – Moscow: Main editorial office of physical and mathematical literature of the publishing house "Nauka", 2006. – 100 p., [in Russian].
5. Sobolev S.L. Uravneniya matematicheskoy fiziki [Equations of mathematical physics]. – Moscow: Main editorial office of physical and mathematical literature of the publishing house "Nauka", 2007. – 432 p., [in Russian].
6. Goloskokov D.P. Analyzing simply supported plates using Maple system // International conference on computer technologies in physical and engineering applications. Saint-Petersburg, 2014. P. 57-58.
7. Kirsanov M.N. Matematika i programmirovaniye v Maple [Mathematics and programming in Maple]: – Moscow: IPR Media, 2020. – 164 p., [in Russian].
8. D'yakonov V.P. Maple 9.5 10 v matematike, fizike i obrazovanii [Maple 9.5 10 in mathematics, physics and education]. – Moscow: SOLON-PRESS, 2017. – 720 p., [in Russian].
9. Govorukhin V.N. Tsibulin V.G. Vvedeniye v Maple. Matematicheskiy paket dlya vsekh [Introduction to Maple. Mathematical package for everyone].– Moscow: Mir, 2017.– 208 p., [in Russian].

Material received on 19.06.24.

Ю.Р. Крахмалева

М.Х. Дулати атындағы Тараз өңірлік университеті, Тараз қ., Казақстан

КОМПЬЮТЕРЛІК МАТЕМАТИКА ЖҮЙЕСІНДЕ БЕССЕЛЬ ФУНКЦИЯЛАРЫ МЕН ТЕҢДЕУЛЕРІ

Аңдатпа. Математикалық физика есептерін шешуде техникалық мәселелерінің өзектілігі жыл сайын артып келеді. Компьютерлік технологияның дамуы және оларды шешудің заманауи әдістерін қолдануға әкеледі. Аналитикалық есептеу жүйелерін пайдалану арқылы тиімді әдіс болып табылады. Ал бұл олардың өнімді жүзеге асуына ықпал етеді. Мақалада Maple бағдарламасындағы Бессель теңдеуінің және оған келтірілетін теңдеулердің жалпы шешімін табу қарастырылып, функциялардың графиктері құрастырылған.

Тірек сөздер: ерекше нүкте, 1-ретті Бессель функциясы, Бессель теңдеуі, 2-ретті Бессель функциясы, асимптотикалық сипаттама.

Ю.Р. Крахмалева

Таразский региональный университет имени М.Х. Дулати, г. Тараз, Казахстан

УРАВНЕНИЕ И ФУНКЦИИ БЕССЕЛЯ В СИСТЕМЕ КОМПЬЮТЕРНОЙ МАТЕМАТИКИ

Аннотация. Актуальность технических вопросов решения задач математической физики с каждым годом растет. Развитие компьютерных технологий приводит к применению современных методов для их решения. Эффективным методом рассматривается применение систем аналитических вычислений, что способствует продуктивному их внедрению. В статье рассматривается нахождение общего решения уравнения Бесселя и приводящихся к нему уравнений в программе Maple, построены графики функции.

Ключевые слова: особая точка, функция Бесселя 1-го рода, уравнение Бесселя, функция Бесселя 2-го рода, асимптотическое поведение.