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CONSTRUCTION OF BASES IN SPACE *M(2,R)* IN COMPUTER MATHEMATICS SYSTEM

Abstract. In finite-dimensional linear spaces, the question of constructing a basis is related to matrix computations, the labor intensity of which increases not only with increasing dimensionality of spaces, but also with taking into account the nature of this space. In this context, we consider an alternative solution of this question in the space of square matrices of the 2nd order in the system of computer mathematics, which has a very rapid development today.

Keywords: linear independence, linear dependence, basis, dimensionality, linear space.

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Introduction. Linear (vector) space is a mathematical structure containing elements – vectors with certain operations of their addition and multiplication by a number – scalar. These operations are referred to the 8 axioms of linear space. A scalar can be an element of real, complex or any other number field. Generalization of the concept "vector" to an element of linear (vector) space not only does not cause displacement of terms, but also allows us to understand or even anticipate a number of results that are valid for spaces of arbitrary nature.

One of the main characteristics of a linear (vector) space are its bases, and together with them the dimensionality of the space [1,2]. The basis of a linear space is defined as the maximum linearly independent finite ordered system of vectors of a linear space such that any vector of this space is linearly expressed through the vectors of the basis. It is known that there are an infinite number of bases in a finite-dimensional linear space [3]. For a space of arbitrary nature, the process of constructing bases has certain nuances, including those of computational nature.

Conditions and methods of research. Let us consider the space of square matrices of the 2nd order $M(2,R)$. The standard basis of this space is the basis $e=(E_1,E_2,E_3,E_4)$:

$$E_1=\begin{pmatrix}1&0\\0&0\end{pmatrix},\ E_2=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ E_3=\begin{pmatrix}0&0\\1&0\end{pmatrix},\ E_4=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

Let us prove that the system of vectors (E_1, E_2, E_3, E_4) is a basis. For this purpose, let us show the linear independence of this system of vectors. According to the definition, a system of vectors (E_1, E_2, E_3, E_4) is linearly independent if there exist such scalars c_1, c_2, c_3, c_4 simultaneously equal to zero that the equality is satisfied:

$$c_1E_1 + c_2E_2 + c_3E_3 + c_4E_4 = O, \tag{1}$$

where O – is a zero matrix of order 2 [4].

Let us write equality (1) in matrix form:

$$c_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

performing operations on the matrices we have:

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2}$$

The corresponding system for finding the unknowns, according to equality (2) is trivial, in which c_1, c_2, c_3, c_4 is already defined:

$$\begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \\ c_4 = 0 \end{cases}. \tag{3}$$

So, the system of vectors (E_1, E_2, E_3, E_4) is linearly independent. Let us show that the system of vectors (E_1, E_2, E_3, E_4) is the maximum linearly independent system. Let's add a vector, for example $E_5 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, to this system and check the fulfillment of equality (1) for the system of vectors $(E_1, E_2, E_3, E_4, E_5)$:

$$c_1E_1 + c_2E_2 + c_3E_3 + c_4E_4 + c_5E_5 = O, \tag{4}$$

where O – is a null matrix of order 2. In matrix form, the last equality is written as follows:

$$\begin{pmatrix} c_1 + c_5 & c_2 + c_5 \\ c_3 + c_5 & c_4 + c_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

To find the unknowns, a system is compiled:

$$\begin{cases} c_1 + c_5 = 0 \\ c_2 + c_5 = 0 \\ c_3 + c_5 = 0 \\ c_4 + c_5 = 0 \end{cases} . \quad (5)$$

System (5) is a homogeneous system of linear algebraic equations in which 4 equations and unknowns. When solving the system, the Kroneckerra-Capelli theorem is applied, according to which the system (5) is joint and indeterminate. This means that the unknowns c_1, c_2, c_3, c_4, c_5 are defined ambiguously. So, the vector system $(E_1, E_2, E_3, E_4, E_5)$ is linearly dependent. It follows that the system of vectors $(E_1, E_2, E_3, E_4, E_5)$ is the basis of space $M(2, R)$, the dimension of space $\dim M(2, R)$is equal to the number of vectors of the basis of space, i.e. 4 .

A linear space can have a large number of bases. Consider an arbitrary system of vectors in space $M(2, R)$, such as the following:

$$F_1 = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix},\ F_2 = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix},\ F_3 = \begin{pmatrix} 3 & 4 \\ 6 & 1 \end{pmatrix},\ F_4 = \begin{pmatrix} -1 & 4 \\ 7 & 9 \end{pmatrix}.$$

Let us examine the system of vectors (F_1, F_2, F_3, F_4) for linear independence. Equality (1) for (F_1, F_2, F_3, F_4) is written as follows:

$$c_1F_1 + c_2F_2 + c_3F_3 + c_4F_4 = O, \quad (6)$$

where O – is a null matrix of order 2. (6) in matrix form:

$$c_1\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} + c_2\begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix} + c_3\begin{pmatrix} 3 & 4 \\ 6 & 1 \end{pmatrix} + c_4\begin{pmatrix} -1 & 4 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

performing matrix operations in the left part of the equality we have:

$$\begin{pmatrix} c_1 + 2c_2 + 3c_3 - c_4 & 2c_1 + 3c_2 + 4c_3 + 4c_4 \\ 3c_1 + 4c_2 + 6c_3 + 7c_4 & 2c_1 + 3c_2 + c_3 + 9c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (7)$$

To determine c_1, c_2, c_3, c_4, we make a homogeneous system of linear algebraic equations:

$$\begin{cases} c_1 + 2c_2 + 3c_3 - c_4 = 0 \\ 2c_1 + 3c_2 + 4c_3 + 4c_4 = 0 \\ 3c_1 + 4c_2 + 6c_3 + 7c_4 = 0 \\ 2c_1 + 3c_2 + c_3 + 9c_4 = 0 \end{cases} \quad (8)$$

As we can see, the system (8) is markedly different from the system (3) and to find the unknowns c_1, c_2, c_3, c_4 it must be solved (the solution is shown in the computer math system Maple). It turns out that in the space $M(2, R)$ to fulfill the definition of the space basis it is necessary to solve 2 homogeneous systems, one of which consists of 4 equations and 4 unknowns, and the other contains 4 equations and 5 unknowns.

Minimize the time-consuming process of calculations, based on the fact that in this or that case certain bases are needed, it is possible to carry out the computational process in the system of computer mathematics, which is Maple [5]. We carry out the computational process in a specialized package *LinalgAlgebra*. At the initial stage we enter the initial data F_1, F_2, F_3, F_4, F_5 and perform matrix operations for vectors F_1, F_2, F_3, F_4 [6], [7] in the left part of equality (6):

```
F1 := Matrix(2, 2, [1, 2, 3, 2]) : F2 := Matrix(2, 2, [2, 3, 4, 3]) : F3 := Matrix(2, 2, [3, 4, 6,
    1]) : F4 := Matrix(2, 2, [-1, 4, 7, 9]) : F5 := Matrix(2, 2, [-1, 1, 2,-1]) :
c1F1 := Multiply(F1, c1); c2F2 := Multiply(F2, c2); c3F3 := Multiply(F3, c3); c4F4
    := Multiply(F4, c4); c5F5 := Multiply(F5, c5);
S1 := MatrixAdd(c1F1, c2F2) : S2 := MatrixAdd(c3F3, c4F4) : S := MatrixAdd(S1, S2); SS
    := MatrixAdd(S, c5F5); Ov := Matrix(2, 2, [0, 0, 0, 0]) :
```

$$S := \begin{bmatrix} c1 + 2\,c2 + 3\,c3 - c4 & 2\,c1 + 3\,c2 + 4\,c3 + 4\,c4 \\ 3\,c1 + 4\,c2 + 6\,c3 + 7\,c4 & 2\,c1 + 3\,c2 + c3 + 9\,c4 \end{bmatrix}$$

The matrix S has the form of the matrix of the left part of the matrix equality (7). We compose a system of equations, which we denote by sys1[5]:

```
sys1 := {S[1, 1] = Ov[1, 1], S[1, 2] = Ov[1, 2], S[2, 1] = Ov[2, 1], S[2, 2] = Ov[2, 2]};
```

$$sys1 := \{c1 + 2\,c2 + 3\,c3 - c4 = 0, 2\,c1 + 3\,c2 + c3 + 9\,c4 = 0, 2\,c1 + 3\,c2 + 4\,c3 + 4\,c4 = 0, 3\,c1 + 4\,c2 + 6\,c3 + 7\,c4 = 0\}$$

To form the matrix A of the system of linear equations, we notice that the coefficients at unknowns c_1, c_2, c_3, c_4 are the elements of matrices F_1, F_2, F_3, F_4, the expanded matrix B of the system, usually consists of matrix A and an attached zero column of free terms:

```
A := Matrix(4, 4, [F1[1, 1], F2[1, 1], F3[1, 1], F4[1, 1], F1[1, 2], F2[1, 2], F3[1, 2], F4[1,
    2], F1[2, 1], F2[2, 1], F3[2, 1], F4[2, 1], F1[2, 2], F2[2, 2], F3[2, 2], F4[2, 2]]);
B := Matrix(4, 5, [F1[1, 1], F2[1, 1], F3[1, 1], F4[1, 1], Ov[1, 1], F1[1, 2], F2[1, 2], F3[1,
    2], F4[1, 2], Ov[1, 2], F1[2, 1], F2[2, 1], F3[2, 1], F4[2, 1], Ov[2, 1], F1[2, 2], F2[2, 2],
    F3[2, 2], F4[2, 2], Ov[2, 2]]);
```

$$A := \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 4 & 4 \\ 3 & 4 & 6 & 7 \\ 2 & 3 & 1 & 9 \end{bmatrix}, B := \begin{bmatrix} 1 & 2 & 3 & -1 & 0 \\ 2 & 3 & 4 & 4 & 0 \\ 3 & 4 & 6 & 7 & 0 \\ 2 & 3 & 1 & 9 & 0 \end{bmatrix}.$$

Then we realize the solution of the system by the Kroneckerr-Capelli theorem, while not missing the case when the rank of the system is less than the number of unknowns (indeterminate system):

```
rA := Rank(A);
rB := Rank(B);
if rA = rB then print ('Sistema_sovmestnaya');
else print ('Sistema_ne_sovmestnaya');
 fi;
 if rA = rB then  rs1 := rA; print ('rs_rang_sistemi');
else print ('rs1_ne_rang_sistemi');
 fi;
n := ColumnDimension(A);
 if rs1 = n  then print ('Sistema_opredelennaya');
B1 := Matrix(4, 1, [Ov[1, 1], Ov[1, 2], Ov[2, 1], Ov[2, 2]]);
 RS1 := LinearSolve(A, B1);
 c11 := RS1[1, 1]; c22 := RS1[2, 1]; c33 := RS1[3, 1]; c44 := RS1[4, 1];
PR := subs(c1 = c11, c2 = c22, c3 = c33, c4 = c44, sys1);
 fi;
 if rs1 < n  then print ('Sistema_ne_opredelennaya');
 B2 := Matrix(4, 1, [Ov[1, 1], Ov[1, 2], Ov[2, 1], Ov[2, 2]]);
 RS2 := LinearSolve(A, B2);
c11 := RS2; c22 := RS2[2, 1]; c33 := RS2[3, 1]; c44 := RS2[4, 1];
PR := subs(c1 = c11, c2 = c22, c3 = c33, c4 = c44, sys1);
 fi;
```

The computation is performed in a loop where the rank of the system *rs1* is equal to the number of unknowns *n*. In this case, the final result is the column matrix $RS1$:

$$RS1 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The elements of the matrix $RS1$ are the unknowns c_1, c_2, c_3, c_4,, which are fixed as $c11, c22, c33, c44$ in the program:

$c11 := 0$

$c22 := 0$

$c33 := 0$

$c44 := 0$

As can be seen, the unknowns c_1, c_2, c_3, c_4 are uniquely defined. We use the conditional operator to check the linear independence of vectors[8]-[10]:

```
if c11 = c22 = c33 = c44 = 0 then print ('Vectori_linenonezavisimie');
else print ('Vectori_linenozavisimie');
fi;
```

print Vectori_linenonezavisimie

Next, we investigate the linear independence of the system of vectors $(F_1, F_2, F_3, F_4, F_5)$ and perform similar matrix operations to form sys2 [6], [7]:

```
sys2 := {SS[1, 1] = Ov[1, 1], SS[1, 2] = Ov[1, 2], SS[2, 1] = Ov[2, 1], SS[2, 2] = Ov[2, 2]};
A1 := Matrix(4, 5, [F1[1, 1], F2[1, 1], F3[1, 1], F4[1, 1], F5[1, 1], F1[1, 2], F2[1, 2], F3[1,
   2], F4[1, 2], F5[1, 2], F1[2, 1], F2[2, 1], F3[2, 1], F4[2, 1], F5[2, 1], F1[2, 2], F2[2, 2],
   F3[2, 2], F4[2, 2], F5[2, 2]]);
B1 := Matrix(4, 6, [F1[1, 1], F2[1, 1], F3[1, 1], F4[1, 1], F5[1, 1], Ov[1, 1], F1[1, 2], F2[1,
   2], F3[1, 2], F4[1, 2], F5[1, 2], Ov[1, 2], F1[2, 1], F2[2, 1], F3[2, 1], F4[2, 1], F5[2, 1],
   Ov[2, 1], F1[2, 2], F2[2, 2], F3[2, 2], F4[2, 2], F5[2, 2], Ov[2, 2]]) ;
```

$$sys2 := \{c1 + 2\,c2 + 3\,c3 - c4 - c5 = 0, 2\,c1 + 3\,c2 + c3 + 9\,c4 - c5 = 0, 2\,c1 + 3\,c2 + 4\,c3 + 4\,c4 + c5 = 0, 3\,c1 + 4\,c2 + 6\,c3 + 7\,c4 + 2\,c5 = 0\}$$

To solve the system *sys2*, the program uses a loop in which the rank of the system *rs2* is less than the number of unknowns n. The column matrix $RSS2$ is the solution matrix of the *sys2* system:

$$RSS2 := \begin{bmatrix} 41\,\_t0_{1,1} \\ -9\,\_t0_{1,1} \\ -9\,\_t0_{1,1} \\ -5\,\_t0_{1,1} \\ \_t0_{1,1} \end{bmatrix}$$

The elements of the matrix are the unknowns c_1, c_2, c_3, c_4, c_5 defined ambiguously:

$$c11 := 41\,\_t0_{1,1}$$
$$c22 := -9\,\_t0_{1,1}$$
$$c33 := -9\,\_t0_{1,1}$$
$$c44 := -5\,\_t0_{1,1}$$
$$c55 := \_t0_{1,1}$$

which allows us to conclude that the vectors $(F_1, F_2, F_3, F_4, F_5)$ are linearly dependent, which is confirmed by the conditional operator[8]-[10]:

print Vectori_linenozavisimie

Research results and discussion. In the program, at the initial stage, the data of space vectors are input, which are investigated to determine the basis of the space. This allows to use the program repeatedly, changing only the input data, which confirms its functionality and automation of the computation process. This technique of baseline construction can be used for baseline construction in the space of square matrices of higher order.

On the example of construction of bases of space $M(2, R)$ the advantages of the modern direction of computer mathematics are demonstrated, which allow to automate the process of calculations, minimize time expenditures on calculations and undeniably confirm their efficiency.

Conclusion. Based on the above, the construction of bases in space $M(2, R)$ by means of computer mathematics systems seems to be one of the rational approaches to solving this issue. The developed mathematical program with a universal algorithm for constructing bases in the Maple system is distinguished by the accuracy and efficiency of calculations**.**

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КОМПЬЮТЕРЛІК МАТЕМАТИКА ЖҮЙЕСІНДЕ *M(2,R)* КЕҢІСТІГІНДЕГІ БАЗИСТЕРДІҢ ҚҰРЫЛУЫ

Аңдатпа. Шекті өлшемді сызықтық кеңістіктерде базисті құру мәселесі матрицалық есептеулермен байланысты, олардың күрделілігі кеңістіктердің өлшемдерінің жоғарылауымен ғана емес, сонымен бірге осы кеңістіктің табиғатын ескере отырып артады. Осыған байланысты, бүгінгі таңда өте қарқынды дамып келе жатқан компьютерлік математика жүйесіндегі 2-ретті квадрат матрицалар кеңістігінде осы мәселенің балама шешімі қарастырылуда.

Тірек сөздер: сызықтық тәуелсіздік, сызықтық тәуелділік, базис, өлшем, сызықты кеңістік.

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ПОСТРОЕНИЕ БАЗИСОВ В ПРОСТРАНСТВЕ *M(2,R)*
В СИСТЕМЕ КОМПЬЮТЕРНОЙ МАТЕМАТИКИ

Аннотация. В конечномерных линейных пространствах вопрос построения базиса связан с матричными вычислениями, трудоемкость которых возрастает не только с повышением размерности пространств, но и с учетом природы этого пространства. В связи с этим, рассматривается альтернативное решение данного вопроса в пространстве квадратных матриц 2-го порядка в системе компьютерной математики, которое имеет весьма стремительное развитие сегодня.

Ключевые слова: линейная независимость, линейная зависимость, базис, размерность, линейное пространство.