IRSTI 27.41.17

N.A. Abiev[1]– *main author*,
M.T. Imanbekova[2]

©

[1]*Candidate of phys.-math. sciences, Associate professor,* [2]*Undergraduate student*

ORCID [1]*https://orcid.org/0000-0003-1231-9396*

M.Kh.DulatyTarazRegional University,
Taraz, Kazakhstan

@ [1]*abievn@mail.ru*

https://doi.org/10.55956/JHOT1010

INTERPRETATION IN MAPLE SYSTEM THE RUNGE PHENOMENON AND ITS ELIMINATING

Abstract. Questions on interpolation of functions with Lagrange polynomials are discussed. The Runge phenomenon is considered, when a sequence of Lagrange polynomials does not converge to an interpolated function uniformly. It is shown that the choice of roots of the Chebyshev polynomials as interpolation nodes provides more precise approximation than the interpolation by equidistant nodes. The Chebyshev polynomials are remarkable in sense that they admit least deviation from zero among all monic polynomials of the same degree. Calculations and drawing graphs were performed using Maple program.

Keywords: interpolation, Lagrange polynomial, equidistant grid, Runge phenomenon, Chebyshev polynomial.

Abiev N.A., Imanbekova M.T. Interpretation in maple system the runge phenomenon and its eliminating // Mechanics and Technology / Scientific journal. – 2021. – No.4(74). – P.102-111. https://doi.org/10.55956/JHOT1010

Introduction. Let $R_n[x]$ denotes the vector space of all real-valued polynomials of degree $\le n$ defined over the set of real numbers R. The vector space of functions defined and continuous on the interval $[a,b]$ we denote $C[a,b]$.

By ∞-norm of a function $f \in C[a,b]$ we mean $\|f\|_\infty := \max\limits_{x\in[a,b]}[f(x)]$.

The interpolation problem can be formulated as follows: for given real numbers $x_0,\dots,x_n$, where $x \ne x_j$ for $i \ne j$, and $y_0,\dots,y_n$ find a polynomial $p_n \in R_n[x]$ such that $p_n(x_i) = y_i$ for $i = 0,\dots,n$.

Methods of studying. We will use the following well-known facts.

Lemma 1. Let $n \ge 0$. Then there exist polynomials $L_k \in R_n[x]$ such that $L_k(x_i) := \begin{cases} 1, & if \ \ i = k, \\ 0, & if \ \ i \ne k, \end{cases}$ for all $k, i \in \{0,\dots,n\}$. Moreover the polynomial $p_n(x) = \sum_{k=0}^{n} L_k(x) y_k$ satisfies the interpolation conditions $p_n(x_i) = y_i$ for $i = 0,\dots,n$.

Proof can be found in [3]. In fact

$$L_k(x)=\prod_{\substack{i=0\\ i\neq 0}}^{n}\frac{x-x_i}{x_k-x_i}. \qquad (1)$$

as follows from Lemma 1.

Lemma 2. Let $n\geq 0$. Assume that $x_0,...,x_n$ are distinct real numbers ($x_i\neq x_j$ for $i\neq j$) and $y_0,...,y_n$ are any real numbers. Then there exists a unique polynomial $p_n\in R_n[x]$ satisfying conditions $p_n(x_i)=y_i$ for $i=0,...,n$.

Proof can be found in [3].

Theorem 1. Let f be a real-valued continuous function defined on a closed interval $[a,b]$, and $x_i\in[a,b]$ be distinct points, $i=0,...,n.$ Then there exists a unique polynomial

$$p_n(x)=\sum_{k=0}^{n} f(x_k)L_k(x) \qquad (2)$$

such that $f(x_i)=p_n(x_i)$ for $i=0,...,n,$ where $L_k(x)$ is given by (1).

Proof follows from Lemmas 1 and 2 putting $y_i := f(x_i), i=0,...,n.$

Definition 1. The polynomial $p_n(x)$ given by (2) is called *the Lagrange interpolation polynomial* of degree n for the function $f(x)$. The distinct interpolation points $x_i\in[a,b]$ are also called *nodes*, $i=0,...,n.$

Estimate of interpolation error. We know that values of $f(x)$ and its Lagrange interpolation polynomial $p_n(x)$ coincide at interpolation points $x_0,...,x_n$. However, $f(x)$ may be quite different from $p_n(x)$ at a point *x* not being an interpolation point, i.e. if $x\neq x_i, i=0,...n.$ Thus the natural question arises: how large may be the difference $f(x)-p_n(x)$ when $x\neq x_i$?

Assume that $n\geq 0$ and $f\in C^{n+1}[a,b]$ (f has the derivative of order $n+1$ continuous on $[a,b]$). The following theorem is well-known (see for example [1,2]).

Theorem 2. Assume that $n\geq 0$ and $f\in C^{n+1}[a,b]$. For the error $\|f-p_n\|$ of interpolation the following estimation holds

$$\max_{x\in[a,b]}|f(x)-p_n(x)|\leq\frac{M_{n+1}}{(n+1)!}\max_{x\in[a,b]}|\omega_{n+1}(x)|, \qquad (3)$$

where $M_{n+1}:=\max_{x\in[a,b]}\left|f^{(n+1)}(x)\right|.$

An important question is whether or not the sequence $\{p_n\}$ of Lagrange polynomials converges to f as $n\to\infty$. If $\lim_{n\to\infty}\frac{M_{n+1}}{(n+1)!}\max_{x\in[a,b]}|\omega_{n+1}(x)|=0$ then $\lim_{n\to\infty}\max_{x\in[a,b]}|f(x)-p_n(x)|=0$. In this case we say that the sequence $\{p_n\}$ of Lagrange polynomials converges to f as $n\to\infty$ uniformly on the interval $[a,b]$.

Runge phenomenon. Runge phenomenon is a problem of oscillation at the edges of an interval that occurs when using polynomial interpolation with polynomials of high degree over a set of equidistant interpolation points. Consider the function $f(x)=\frac{1}{1+5x^2}$, $x\in[-1,1]$. Then the sequence of Lagrange polynomials does not converge to $f(x)$ uniformly if interpolation points are equally spaced as $x_i=-1+\frac{2i}{n}$.

Interpolation accuracy improving using zeros of Chebyshev polynomials. On [−1, 1] consider functions

$$T_n(x):=\cos(n\arccos x). \qquad (4)$$

It is easy to observe that $T_n(x)$ is a polynomial of degree n defined on $[-1,1]$. Indeed $T_1(x)=x$ at $n=1$ and $T_2(x)=\cos(2\arccos x)=2(\arccos x)-1=2\cos^2(\arccos x)-1=2x^2-1$ at $n=2$.

Using equality $\cos(n+1)\theta+\cos(n-1)\theta=2\cos\theta\cos(n\theta)$ and assuming $\theta:=\arccos x$ we get the recurrent formula

$$T_{n+1}(x)=2xT_n(x)=2xT_n(x)-T_{n-1}(x), \quad (5)$$

showing that $T_n(x)$ is indeed a polynomial of degree n.

Extrema of Chebyshev polynomials. It is clear by definition that $|T_n(x)|\le 1$ for all $x\in[-1,1]$ and all $n\ge 0$. Moreover, $\|T_n\|_\infty=\max_{x\in[-1,1]}|T_n(x)|=1$.

The maximum value 1 of $T_n(x)$ is achieved at $n+1$ distinct points

$$x_m:=\cos\frac{m\pi}{n}, \qquad (6)$$

$m=0,...,n.$ This follows from (4) and the fact that $|\cos(t)=1|$ has solutions $t=\pi m$.

Zeros of Chebyshev polynomials. $T_n(x)$ has n distinct roots

$$\xi_i=\cos\frac{2i+1}{2n}\pi \qquad (7)$$

in the interval [−1, 1], where $i=0,...,n-1$.

Minimality property of Chebyshev polynomials. A polynomial of degree n whose leading coefficient (the coefficient of the leading monomial x^n) is equal to 1, is called a *monic* polynomial of degree n . It is easy to observe from (5) that the leading coefficient of $T_n(x)$ equal to 2^{n-1} . Then we can construct monic polynomials $\tilde{T}_n(x):=\frac{1}{2^{n-1}}T_n(x)$. Clearly, $\|\tilde{T}_n\|_\infty=\frac{1}{2^{n-1}}$.

Lemma 3. Suppose that $n\ge 0$. Among all monic polynomials of degree n the polynomial $\tilde{T}(x)$ has the smallest ∞-norm on the interval [−1, 1], i.e.

$$\max_{x\in[-1,1]}\left|\tilde{T}_n(x)\right|\le\max_{x\in[-1,1]}|P_n(x)|$$

for any monic polynomial $P_n(x)$ of degree *n* defined on [−1, 1].

Proof (see [3]). Let be an arbitrary monic polynomial of degree n defined on [−1, 1]. We claim that $\left\|\tilde{T}_n\right\|_\infty \le \|P_n\|_\infty$.

Suppose by contrary that $\|P_n\|_\infty < \left\|\tilde{T}_n\right\|_\infty$. Since $\left\|\tilde{T}_n\right\|_\infty = 2^{-(n-1)}$ we have $|P_n(x)| < 2^{-(n-1)}$ for all [−1, 1]. Moreover, $\left|\tilde{T}_n(x)\right| = 2^{-(n-1)}$ at the points given in (6). Hence admitting the sign of $\tilde{T}_n(x)$ on the points $x_0,...,x_n$, the polynomial $Q_{n-1}(x) := \tilde{T}_n(x) - P_n(x)$ will alternate its signs $n+1$ times. So Q_{n-1} admits at least *n* sign changes. Hence Q_{n-1} should have at least *n* roots. But Q_{n-1} was a polynomial of degree $n-1$. We get a contradiction. Lemma 3 proven.

Lemma 3 suggests that the interpolation points (nodes) $x_0,...,x_n$ should be taken as the zeros of the Chebyshev polynomial T_{n+1}, for them will have the smallest ∞-norm on the interval [1,1] among all monic polynomials. Thus the nodal function ω_{n+1} will be

$$\omega_{n+1}(x) = 2^{-n} T_{n+1}(x) = (x-\xi_0)...(x-\xi_n) \qquad (8)$$

with

$$\xi_i := \cos\frac{i+\frac{1}{2}}{n+1}\pi,\ i = 0,...,n \quad (9)$$

according to (7). Recall the error estimation $\|f - p_n\|$ of interpolation given in Theorem 2. Assume that $n \ge 0$ and $f \in C^{n+1}[-1,1]$. Since (8) implies $\|\omega_{n+1}\| = 2^{-n}$, the estimation (3) takes the simple form

$$\|f - p_n\| = \max_{x\in[-1,1]}|f(x) - p_n(x)| \le \frac{M_{n+1}}{(n+1)!}2^{-n} \qquad (10)$$

ifzeros $\xi_0,...,\xi_n$ (see (9)) of the Chebyshev polynomial T_{n+1} are chosen on the interval [−1, 1] as interpolation points, where $M_{n+1} := \max_{x\in[-1.1]}\left|f^{(n+1)}(x)\right|$.

According such a property Chebyshev polynomials are called polynomials *least deviatingfrom zero*.

Obtained results. The following results were obtained using Maple [4] to interprate theoretical results mentioned above.

Being convenient for calculations by hand formula (1) is not effective for computer programming. Introducing the nodal function formula $\omega_{n+1}(x) = \prod_{i=0}^{n}(x - x_i)$ (1) can be written in the form

$$L_k(x) = \frac{\omega_{n+1}(x)}{(x - x_k)\omega'_{n+1}(x_k)}.$$

Basing on this formula (1) a Maple program was prepared by the authors that constructs Lagrange polynomials for any function f and any n and draws their graphs.

Example 1. Consider the function $f(x) = \cos(x) + \sin(2x)$ defined on the interval $[0, 2\pi]$. Assume that interpolation points are equidistant: $x_i = \frac{2\pi}{n} i$.

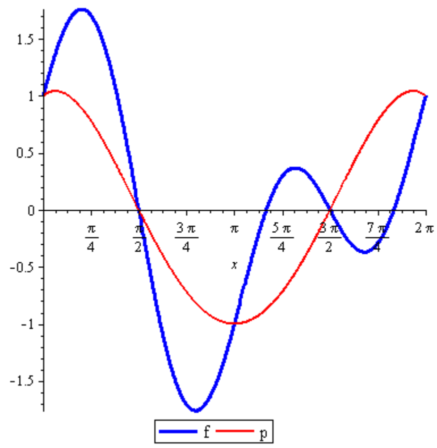
Take $n = 4$. As Maple calculations show

$$p_4(x) = -\frac{8}{3\pi^4}x^4 + \frac{32}{3\pi^3}x^3 - \frac{34}{3\pi^2}x^2 + \frac{4}{3\pi}x + 1.$$

Note that expressions for $p_n(x)$ are very huge for increasing values of *n*. For example,

$$p_8(x) = -0.000020170538x^8 - 0.00810359652x^7 + 0.1851451483x^6 - $$
$$- 1.589416061x^5 + 6.44694644x^4 - 12.17806018x^3 + 8.41909159x^2 - $$
$$- 0.77199978x + 1.$$

In Figure 1 you can see graphs of $f(x)$ and $p_n(x)$ drawn by our program at $n = 4$. The corresponding graphs at $n = 8$ and $n = 10$ are depicted in Figure 2.



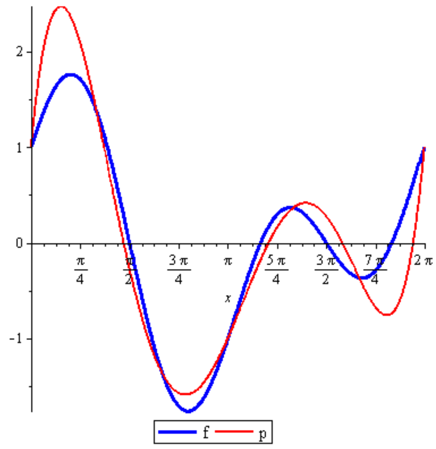


Figure 1. The function $f(x) = \cos(x) + \sin(2x)$ and its Lagrange polynomials for *n*= 4 (theleft panel) and *n* = 6 (the right panel)

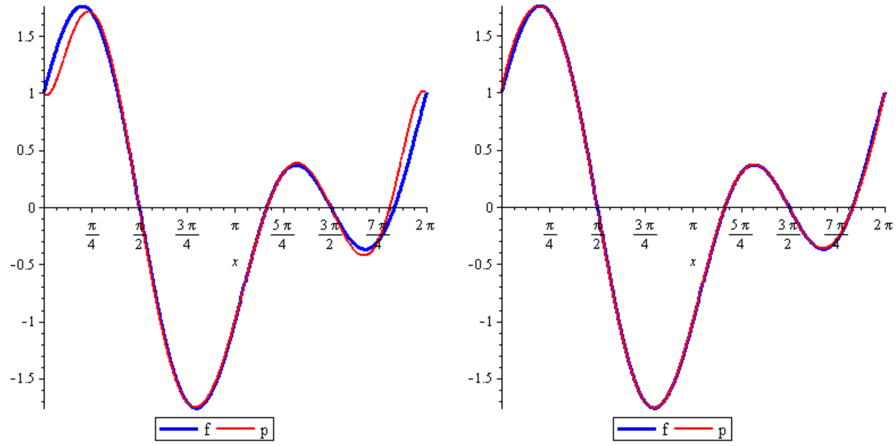


Figure 2. The function $f(x)=\cos(x)+\sin(2x)$ and its Lagrange polynomials for n = 8 (theleft panel) and n = 10 (the right panel)

In Figures 3 and 4the Runde phenomenon is illustrated. You can see the graphs of $f(x)=\dfrac{1}{1+5x^2}$ and $p_n(x)$ drawn by using of our program on equidistant gridat $n=10, n=20, n=40$ and $n=50$.

The discovery of Runge was important because it shows that going to higher degrees does not always improve accuracy.

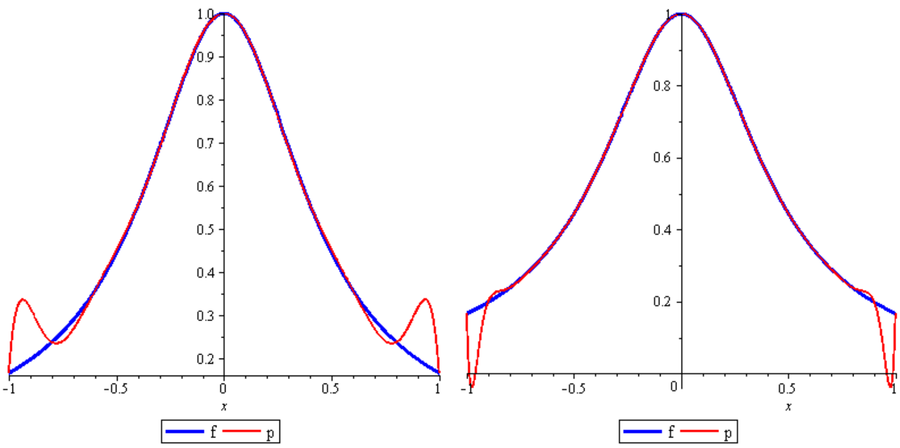


Figure 3. Lagrange interpolation of $f(x)=\dfrac{1}{1+5x^2}$ using equidistant nodes at $n=10$ (the left panel) and $n=20$ (the right panel)

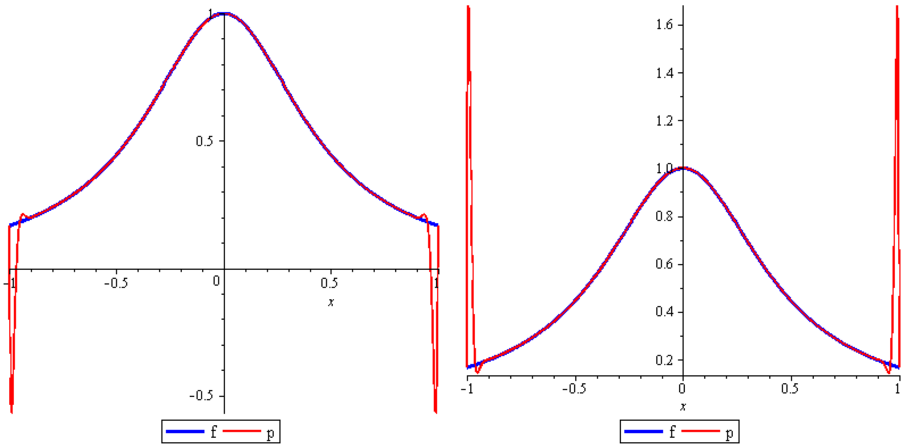


Figure 4. Lagrange interpolation of $f(x)=\dfrac{1}{1+5x^2}$ using equidistant nodes at $n=40$ (the left panel) and $n=50$ (the right panel)

In Figure 5 the monicChebyshevpolynomials $\tilde{T}_n(x)$ are constructing by Maple at $n=3,4,5$.

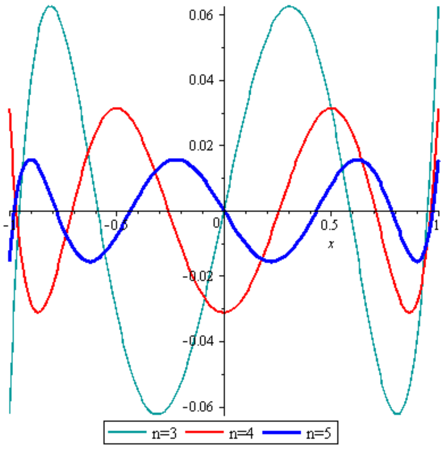


Figure 5. Polynomials $\tilde{T}_n(x)$ at $n=3,4,5$.

Example 2. *Revisiting Runge phenomenon***.** Choose the zeros of Chebyshev polynomials given by (9) as interpolation nodes. Then on the interval $[-1,1]$ the Lagrange polynomial

$$p_n(x)=\sum_{k=0}^{n} f(\xi_k)L_k(x)$$

approximates

$$f(x)=\frac{1}{1+5x^2}, x\in[-1,1]$$

more precisely than the Lagrange polynomial on equidistant nodes as follows from the estimate (10).

This fact is confirmed by our calculations in Maple. The behavior of Lagrange polynomials are depicted in Figures 6 and 7. Compare them with Figures 3 and 4.

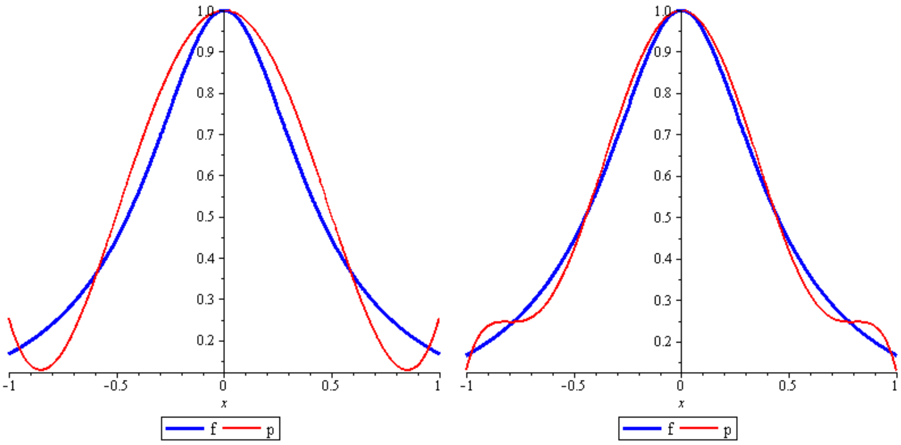


Figure 6. Lagrange interpolation of $f(x)=\dfrac{1}{1+5x^2}$ using Chebyshev nodes at $n=4$ (the left panel) and $n=6$ (the right panel)

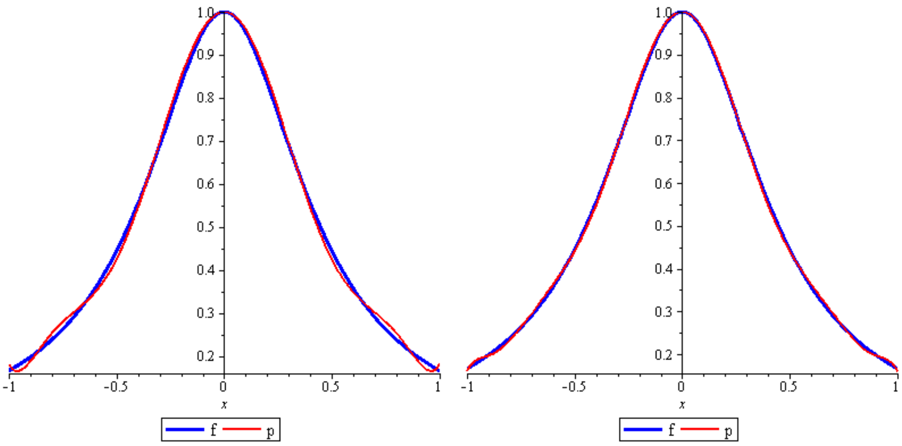


Figure 7. Lagrange interpolation of $f(x)=\dfrac{1}{1+5x^2}$ using Chebyshev nodes at $n=8$ (the left panel) and $n=10$ (the right panel)

Program codes. Here are main program codes in Maple constructing Lagrange polynomials on equidistant grids in the case of Example 1:

> **for** i **from** 0 **to** n **do** $u_i := a + \dfrac{b-a}{n}\cdot i$ **end do**:

> $\omega := 1$:

> **for** i **from** 0 **to** n **do** $\omega := \omega\cdot\left(x-u_i\right)$ **end do**:

> $\tau := \mathit{diff}(\omega, x)$:

$> p_n := 0:$

$>$ **for** i **from** 0 **to** n **do** $p_n := \left(p_n + f(u_i) \cdot \dfrac{\omega}{(x - u_i) \cdot subs(x = u_i, \tau)}\right):$ **end do**:

Graphical commands:

$>$
$g := plot(f(x), x = a\,..b, numpoints = 1500, color = blue, thickness = 3, legend = "f"):$

$> l := plot(p_n, x = a\,..b, numpoints = 1500, color = red, legend = "p"):$

$> display([g, l], scaling = unconstrained)$

In the case of Example 2 for nodes as Chebyshev polynomials' roots we offer the additional command for node constructions:

$>$ **for** i **from** 0 **to** n **do** $u_i := simplify\left(\cos\left(\dfrac{\pi}{2\cdot(n+1)} + \dfrac{\pi\cdot i}{n+1}\right)\right)$ **end do**:

Final (supplementary) remarks (the general case $x \in [a,b]$). The linear function $x = \frac{1}{2}[(b-a)t + b + a]$ maps $[-1,1]$ onto $[a,b]$. Then the nodes (9) in $[-1,1]$ are transformed into nodes $\mu_i = \frac{1}{2}[(b-a)\xi_i + b + a]$, $i = 0,\dots,n$ in $[a,b]$.

Theorem 3 (see [3]). Let $n \geq 0$ and $f \in C^{n+1}[a,b]$. Assume that the points $\mu_0,\dots\mu_n$ are chosen as interpolation nodes. Then the error $\|f - p_n\|$ of interpolation has the following estimation $\|f - p_n\| \leq \dfrac{M_{n+1}}{(n+1)!}\left(\dfrac{b-a}{2}\right)^{n+1} 2^{-n}$, where $M_{n+1} := \max_{x\in[a,b]}\left|f^{n+1}(x)\right|$.

It is easy to observe that (10) is a partial case of (11) at $a = 0$ and $b = 1$.

Conclusion. Unfortunately, uniform convergence of the Lagrange sequence $\{p_n\}$ to f as $n \to \infty$ on the interval $[a,b]$ need not to be true in general if equidistant nodes are chosen. Sometimes the sequence $M_{n+1} \max_{x\in[a,b]}|\omega_{n+1}(x)|$ may tend to ∞ as $n \to \infty$ faster than the sequence $\dfrac{1}{(n+1)!}$ tends to 0. We met this fact on the example of Runge phenomenon. Note also that the choice of roots of Chebyshev polynomials provides the best approximation.

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Material received 02.12.21.

Н.А.Абиев, М.Т. Иманбекова

М.Х. Дулати атындағы Тараз өңірлік университеті, Тараз, Қазақстан

РУНГЕ ҚҰБЫЛЫСЫН ЖӘНЕ ОНЫ ЖОЮДЫ MAPLE ЖҮЙЕСІНДЕ ИНТЕРПРЕТАЦИЯЛАУ

Аңдатпа. Жұмыста функцияларды Лагранж көпмүшеліктерімен жуықтау сұрақтары талқыланады. Интерполяцияланатын функцияға Лагранж көпмүшеліктері тізбегі бірқалыпты жинақталмаған жағдайдағы Рунге құбылысы қарастырылған. Интерполяция түйіндері ретінде Чебышев көпмүшеліктері түбірлерін таңдау бірдей қашықтықтағы түйіндердегі интерполяциялауға салыстырмалы дәлірек жуықтауды қамтамасыз ететіні көрсетілген. Чебышев көпмүшеліктері бірдей дәрежедегі келтірілген көпмүшеліктердің ішінен нөлден ең аз ауытқуымен белгілі. Есептеулер мен графиктерді тұрғызу Maple бағдарламасы көмегімен жүзеге асырылған.

Тірек сөздер: интерполяция, Лагранж көпмүшелігі, бірқалыпты тор, Рунге құбылысы, Чебышев көпмүшелігі.

Н.А. Абиев, М.Т. Иманбекова

Таразский региональный университет им. М.Х. Дулати,Тараз, Казахстан

ИНТЕРПРЕТАЦИЯ В СИСТЕМЕ MAPLE ЯВЛЕНИЯ РУНГЕ И ЕГО УСТРАНЕНИЯ

Аннотация. В работе обсуждаются вопросы интерполяции функций многочленами Лагранжа. Рассмотрено явление Рунге, когда последовательность многочленов Лагранжа несходится равномерно к интерполируемой функции. Показано, что выбор корней многочленов Чебышева в качестве узлов интерполирования обеспечивает более точное приближение, чем интерполирование по равноотстающим узлам. Многочлены Чебышева замечательны тем, что они наименее отклоняются от нуля среди всех приведенных многочленов одинаковой степени. Вычисления и построения графиков проведены с помощью Maple программы.

Ключевые слова: интерполяция, многочлен Лагранжа, равномерная сетка, явление Рунге, многочлен Чебышева.